

On fresh sets in iterations of Prikry-type forcing notions

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April 21, 2025

Abstract

We examine the existence (and mostly non-existence) of fresh sets in commonly used iterations of Prikry-type forcing notions. Results of [4] are generalized. As an application, a question of a referee of [10] is answered, as well as a question of Goldberg regarding the Cover and Approximation properties. In addition, preservation of stationary sets is addressed.

1 Introduction

Let P be a forcing notion and $G \subseteq P$ its generic subset. Suppose that W is a κ -complete ultrafilter over κ in $V[G]$. Let $U = W \cap V$. In general, U need not be in V . However, by J. Hamkins [7], if P has a gap below κ , i.e. for some $\beta < \kappa$, $P = R * \underset{\sim}{Q}$ such that $|R| < \beta$ and $\underset{\sim}{Q}$ is $\beta + 1$ -strategically closed, then $U \in V$. One of the main techniques was the analysis of fresh sets in forcings extensions with P .

Following Hamkins, a subset Z of λ is called *fresh* (over V) if for every $\alpha < \lambda$, $Z \cap \alpha \in V$, but $Z \notin V$.

In [4] the following was shown:

Theorem 1.1 *$U \in V$, provided that–*

1. *all cardinals of V in the interval $[\kappa, (2^\kappa)^V]$ are preserved,*
2. *no fresh subsets are added to cardinals λ , $\kappa \leq \lambda \leq (2^\kappa)^V$.*

*We are grateful to participants of TAU Set Theory seminar and in particular to Menachem Magidor and to Omer Ben-Neria for their comments and remarks. We would like to thank the referee for reading the paper and offering improvements and useful remarks. The work of the first author was partially supported by ISF grant No. 882/22.

So, fresh sets are relevant for understanding whether $U \in V$.

Structural properties of the forcing being considered may rule out the possibility that fresh subset are added to certain cardinals. J. Hamkins [7] proved that if a forcing has a gap below κ , then no fresh subsets of a cardinal μ whose cofinality is above the gap are added. Another example in this spirit is the following Lemma from [4]:

Lemma 1.2 *Let κ be measurable. Let $V \subseteq V[G]$ be a forcing extension that preserves κ^+ . Assume that some normal measure $U \in V$ on κ concentrates on the set $\{\alpha < \kappa: 2^\alpha = \alpha^+\}$ and extends to a normal measure $W \in V[G]$ on κ . Then no fresh subsets are added to κ in the extension $V \subseteq V[G]$.*

We aim to extend Hamkins' approach to a broader class of forcing notions, focusing primarily on iterations of Prikry-type forcings. Lemma 1.2 can be used, for example, to prove that such iterations - of length which is a measurable cardinal κ - do not introduce any fresh subsets of κ .¹ Additional results concerning nonexistence of fresh subsets of κ^+ were provided in [4] as well². These results suffice to analyze the structure of normal measures on κ after performing iterations of Prikry-type forcings. This analysis has been done in a series of papers [4], [5],[9], [6].

Continuing the series of papers, the present work focuses mainly on fresh sets. We will generalize the aforementioned results from [4], and connect them to new results regarding preservation of stationarity. The motivating questions that this paper answers are the following:

Question 1.3 Let κ be an inaccessible cardinal. Which stationary subsets $S \subseteq \kappa$ remain stationary after forcing with an iteration of Prikry-type forcings of length κ ?

Question 1.4 Assume that W is a κ complete ultrafilter in the generic extension, after forcing with an iteration of Prikry-type forcings of length κ . Assume that W concentrates on a ground model set. Is it true that W lifts a κ -complete ultrafilter of the ground model?³

Question 1.5 Let κ be an inaccessible. Can iterations of length κ of Prikry-type forcings add fresh subsets to κ ? (note that Lemma 1.2 doesn't apply if κ is not measurable).

¹Some relatively mild assumptions are needed to be imposed on the forcing in order to deduce that. We will provide below a list of sufficient conditions (see subsection 1.2 below). See also [4, Corrolary 4.7] for a detailed proof.

²See Corollary 4.9, Lemma 4.10 and Lemma 4.11 in [4].

³Note that Theorem 1.1 applies only to the case where W concentrates on κ . A central example to be considered here is whether κ -complete ultrafilters on $(P_\kappa(\lambda))^{V[G]}$ (for $\lambda > \kappa$) restrict to ultrafilters of V , provided that they concentrate on $(P_\kappa(\lambda))^V$.

Question 1.6 Can the nonstationary support iteration of Prikry-type forcings (a definition of the nonstationary support will be given below) add new measurable cardinals?⁴

Question 1.7 (Goldberg) Let κ be an inaccessible. Does iterations of length κ of Prikry-type forcings have Hamkins' κ -cover and approximation properties?⁵

1.1 Framework

Let us describe the framework of this paper. Throughout the entire paper,

$$\langle P_\alpha, \mathcal{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$$

is an iteration of length κ of Prikry-type forcings, taken with either Magidor (full support), nonstationary or Easton support. This means that, for every $\alpha < \kappa$,

$$\Vdash_{P_\alpha} \text{''}\langle \mathcal{Q}_\alpha, \leq_{\mathcal{Q}_\alpha}, \leq_{\mathcal{Q}_\alpha}^* \rangle \text{ is a Prikry-type forcing notion''}$$

and conditions in P_α , for a given $\alpha \leq \kappa$, are sequences $p = \langle \mathcal{p}(\beta) : \beta < \alpha \rangle$ such that, for every $\beta < \alpha$, $p \restriction \beta \in P_\beta$ and $p \restriction \beta \Vdash \mathcal{p}(\beta) \in \mathcal{Q}_\beta$. Additional requirements might be imposed on the set $\text{supp}(p) = \{\beta < \alpha : p \restriction \beta \Vdash \mathcal{p}(\beta) \neq \emptyset_{\mathcal{Q}_\beta}\}$, depending on the chosen support for the iteration; this point will be further explained below.

Whenever $p = \langle \mathcal{p}(\beta) : \beta < \alpha \rangle, q = \langle \mathcal{q}(\beta) : \beta < \alpha \rangle \in P_\alpha$, we say that q extends p , and denote $q \geq_{P_\alpha} p$, if for every $\beta < \alpha$, $q \restriction \beta \geq_{P_\beta} p \restriction \beta$, $q \restriction \beta \Vdash \mathcal{p}(\beta) \leq_{\mathcal{Q}_\beta} \mathcal{q}(\beta)$, and for all but finitely many $\beta \in \text{supp}(p)$, $q \restriction \beta \Vdash \mathcal{q}(\beta) \geq_{\mathcal{Q}_\beta}^* \mathcal{p}(\beta)$. The extension $q \geq p$ is called direct if for all $\beta \in \text{supp}(p)$, $q \restriction \beta \Vdash \mathcal{q}(\beta) \geq_{\mathcal{Q}_\beta}^* \mathcal{p}(\beta)$, in which case we denote $q \geq^* p$.

We refer the reader to [3] for a discussion about the full and Easton support iterations, and to [1] for a discussion about nonstationary support iterations.

We will also include the following assumptions:

1. GCH.
2. κ is an inaccessible cardinal.
3. For every $\beta < \kappa$, $\langle \mathcal{Q}_\beta, \leq_{\mathcal{Q}_\beta}^* \rangle$ is forced to be $|\beta|$ -closed.

Note that for a singular β this implies $|\beta|^+$ -closure.

⁴This question was raised by the referee of [10]. Regarding the Full and Easton support iterations, a negative answer is already known, see [3].

⁵For a definition of the properties, see section 6.

4. For every $\beta < \kappa$, $|\mathcal{Q}_\beta| < \kappa$.
5. If the Easton support is used, then for every $p \in P_\kappa$ and every inaccessible $\alpha \leq \kappa$, $\alpha > |\text{supp}(p) \cap \alpha|$, provided that for every $\beta < \alpha$, $|P_\beta| < \alpha$.
6. If the non-stationary support is used, then for every $p \in P_\kappa$ and every inaccessible $\alpha \leq \kappa$, $\text{supp}(p) \cap \alpha$ is non-stationary in α , provided that for every $\beta < \alpha$, $|P_\beta| < \alpha$.

1.2 Structure of the paper

We conclude the introduction by summarizing the structure and the main theorems of the paper.

- In Section 2 we deal with preservation of stationarity. The main result is Theorem 2.2, in which we prove that, for $S \subseteq \kappa$, its stationarity is preserved after forcing with P_κ with an Easton or nonstationary support; we also provide a sufficient condition for preservation of S when the full support is taken. This answers Question 1.3.
- In Sections 3-5 we address fresh subsets, providing a negative answer to Question 1.5 under relatively mild additional assumptions on the forcing P_κ . The proof methods depend on the chosen support: in section 3 we deal with the nonstationary support (Theorem 3.1); in section 4 we deal with the Easton support (Theorem 4.1 under the additional assumption that κ is Mahlo, and see Theorem 4.5 for a version without it); in section 5 we deal with the full support (Theorem 5.1).
In addition, we provide in section 3 a negative answer to Question 1.6, see Theorem 3.5.
- In Section 6, we address Hamkins' Cover and Approximation properties. We show in Theorem 6.6 that P_κ satisfies the κ -Cover and κ -Approximation properties (as long as it satisfies the mild assumptions needed for the proofs that no fresh subsets are added, given in Sections 3-5), answering question 1.7 above. As an application, an affirmative answer to Question 1.4 is given in Corollary 6.7. This generalizes Theorem 1.1 in the context of iterations of Prikry-type forcings.

2 On preservation of stationarity in Prikry-type extensions

Let κ be an inaccessible cardinal and let S be a stationary subset of κ . Suppose that $\langle P_\alpha, \mathcal{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is an Easton or a full (Magidor) or a non-stationary support iteration of Prikry-type forcing notions as in the introduction. Let G_κ be a generic subset of P_κ .

We would like to address here a question *whether S remains stationary in $V[G_\kappa]$* .

Remark 2.1 1. Note that we can assume that κ is a regular cardinal in V , since otherwise it is possible to replace it by $\text{cof}(\kappa)$.

2. Our main interest will be in situations where κ is an inaccessible cardinal in $V[G]$.
3. If κ is a Mahlo cardinal, then Easton support iterations satisfy κ -c.c. and so preserves stationary subsets of κ .
4. If a full support iteration of Prikry forcings is used, then the set of former measurables which changed their cofinality will be non-stationary, as witnessed by the regressive function which maps each one of them to the first element in its Prikry sequence.

Theorem 2.2 *Assume GCH. Let κ be an inaccessible cardinal. Let $P = P_\kappa$ be as in the introduction. Let $G \subseteq P_\kappa$ be generic over V . Then κ is inaccessible in $V[G]$, and–*

1. *If $S \subseteq \kappa$ is stationary in V and consists of singulars, then S remains stationary in $V[G]$.*
2. *If the Easton or nonstationary support is used, then P_κ preserves stationary subsets of κ .*
3. *If the full support is used, $S \subseteq \kappa$ is stationary in V , and–*
 - (a) *For every $\alpha \in S$, $\langle \mathcal{Q}_\alpha, \leq^* \rangle$ is $|\alpha|^+$ -complete.*
 - (b) *For every $\alpha < \kappa$, $\mathcal{Q}_\alpha$ has the property that for every $p, q, r \in \mathcal{Q}_\alpha$, if $p, q \geq^* r$, then there is $t \in \mathcal{Q}_\alpha$ such that $t \geq^* p, q$.*

Then S remains stationary in $V[G]$.

The inaccessibility of κ in $V[G]$ is proved in 2.6. Point 1 of the Theorem is proved in 2.7. Point 2 of the Theorem is proved in Theorems 2.9 and 2.11. Point 3 of the Theorem is proved in 2.10.

Let us start with the following lemma which is a weak form of a strong Prikry condition:

Lemma 2.3 *Let $D \subseteq P_\kappa$ be a dense open and let $p \in P_\kappa$. Then there are $\alpha < \kappa$ and $q \geq^* p$ such that for every $r \in P_\alpha$, $r \geq_{P_\alpha} q \restriction \alpha$ there is $r' \geq_{P_\alpha} r$ such that $r' \restriction q \setminus \alpha \in D$.*

Proof. Let D and $p = \langle \underset{\sim}{p}(\gamma) \mid \gamma < \kappa \rangle$ be as in the statement of the lemma.

We prove the lemma for the full support iteration. The arguments for the Easton and for the non-stationary support iterations are very similar, only coordinates in supports should be considered.

Suppose that the conclusion of the lemma fails.

We define by recursion, for each $\beta < \kappa$, a condition–

$$p^\beta = \langle \underset{\sim}{p}^*(\gamma) \mid \gamma < \beta \rangle \frown \langle \underset{\sim}{p}(\gamma) \mid \beta \leq \gamma < \kappa \rangle$$

so that $p^\beta \restriction \beta \Vdash \neg \sigma_\beta$, where

$$\sigma_\beta \equiv \exists t \in P_\kappa \setminus \beta, t \geq^* p \restriction \beta \exists r \in \mathcal{G}_\beta \quad r \restriction t \in D.$$

For the first stage, we have–

$$\sigma_1 \equiv \exists t \in P_\kappa \setminus 1, t \geq^* p \restriction 1 \exists r \in \mathcal{G}_1 \quad r \restriction t \in D.$$

Q_0 satisfies the Prikry condition, so there is $p_0^* \geq^* p_0$ which decides σ_1 . If $p_0^* \Vdash \sigma_1$, then $p_0^* \restriction \underset{\sim}{t}$ will be as desired. So, assume that $p_0^* \Vdash \neg \sigma_1$, and continue.

The successor step is similar to the first stage above.

For limit steps, suppose that β is a limit ordinal. Let us show that–

$$p^\beta = \langle \underset{\sim}{p}^*(\gamma) \mid \gamma < \beta \rangle \frown \langle \underset{\sim}{p}(\gamma) \mid \beta \leq \gamma < \kappa \rangle$$

is as desired, i.e. $p^\beta \restriction \beta \Vdash \neg \sigma_\beta$. Suppose otherwise, then there is $r = \langle \underset{\sim}{r}(\gamma) \mid \gamma < \beta \rangle \in P_\beta$ such that $r \geq p^\beta \restriction \beta$ and $r \Vdash \sigma_\beta$. Extend it, if necessary, so that for some $\underset{\sim}{t}$

$$r \restriction \underset{\sim}{t} \geq^* p \restriction \beta \text{ and } r \restriction \underset{\sim}{t} \in D.$$

By the definition of order on $\mathcal{P}_\beta$, there is $\beta^* < \beta$ such that for every γ , $\beta^* \leq \gamma < \beta$,

$$r \restriction \gamma \Vdash \check{r}(\gamma) \geq^*_{\check{p}} \check{p}^*(\gamma).$$

Consider a P_{β^*} -name

$$\check{t}' = \langle \check{r}(\gamma) \mid \beta^* \leq \gamma < \beta \rangle \check{\smallfrown} \check{t}.$$

Then

$$r \restriction \beta^* \Vdash \check{t}' \geq^* p \restriction \beta^* \text{ and } r \restriction \beta^* \check{\smallfrown} \check{t}' \in D.$$

But $r \restriction \beta^* \geq p^{\beta^*} \restriction \beta^* \Vdash \neg \sigma_{\beta^*}$. Contradiction.

This completes the construction.

Consider $p(\alpha) = \langle \check{p}^*(\gamma) \mid \gamma < \kappa \rangle$. Pick some $r \geq p^\kappa$ in D . Now we obtain a contradiction as in the limit stage above.

□

Assuming that κ is measurable in the ground model, a stronger version of Lemma 2.3 can be proved:

Lemma 2.4 *Assume that κ is measurable in V and U is a normal measure on κ . Let $D \subseteq P_\kappa$ be a dense open and let $p \in P_\kappa$. Then there are $\alpha < \kappa$ and $q \geq^* p$, such that $q \restriction \alpha = p \restriction \alpha$, and for every $r \in P_\alpha, r \geq_{P_\alpha} p \restriction \alpha$ there is $r' \geq_{P_\alpha} r$ such that $r' \check{\smallfrown} q \restriction \alpha \in D$. Moreover, for every $X \in U$, α, q above can be chosen such that $\alpha \in X$.*

Proof. Let σ_β be as in lemma 2.3. If there exists $\beta \in X$ such that $p \restriction \beta \Vdash \sigma_\beta$, we are done.

Assume otherwise. For every $\beta \in X$ there exists $r_\beta \geq p \restriction \beta$ such that $r_\beta \Vdash \neg \sigma_\beta$. For each such limit β , there exists $\beta' < \beta$ such that $r_\beta \restriction \beta' \Vdash r_\beta \restriction \beta' \geq^* p \restriction [\beta', \beta)$. The function $\beta \mapsto \beta'$ is regressive, and thus there exist a set $A \in U$ and $\beta^* < \kappa$ such that for every $\beta \in A$, $r_\beta \restriction \beta^* \Vdash r_\beta \restriction \beta^* \geq^* p \restriction [\beta^*, \beta)$. Since $|P_{\beta^*}| < \kappa$, we can shrink $A \in U$ further, and assume that there exists $r^* \in P_{\beta^*}$, such that, for every $\beta \in A$, $r_\beta \restriction \beta^* = r^*$.

Then r^* has the following property: for every $\beta \in A$ there exists $s(\beta) \geq^* p \restriction [\beta^*, \beta)$ such that $r^* \check{\smallfrown} s(\beta) \Vdash \neg \sigma_\beta$. Now apply ineffability: we can find $A^* \subseteq A$, $A^* \in U$, and a P_{β^*} -name for a condition $s^* = [\beta \mapsto s(\beta)]_U \in P \setminus \beta^*$, such that $r^* \Vdash s^* \geq^* p \restriction \beta^*$, and, for every $\beta \in A^*$,

$$r^* \check{\smallfrown} s^* \restriction [\beta^*, \beta) = r^* \check{\smallfrown} s(\beta) \Vdash \neg \sigma_\beta.$$

Finally, pick some $q \geq r^* \check{\smallfrown} s^*$, $q \in D$. Let $\beta \in A^* \setminus \beta^* + 1$ be such that–

$$q \restriction \beta \Vdash q \restriction \beta \geq^* s^* \restriction \beta$$

and in particular,

$$q \restriction \beta \Vdash q \restriction \beta \geq^* p \restriction \beta, \text{ and } q \restriction \beta \restriction q \restriction \beta \in D$$

and thus $q \restriction \beta \Vdash \sigma_\beta$; however, since $\beta \in A^*$,

$$q \restriction \beta \geq r^* \restriction s^* \restriction [\beta^*, \beta) \Vdash \neg \sigma_\beta$$

which is a contradiction.

□

Lemma 2.5 *Let $p \in P_\kappa$ be a condition, and assume that ζ a P_κ -name for an ordinal. Then there exists $q \geq^* p$ and a set of ordinals $A \in V$ with $|A| < \kappa$ such that $q \Vdash \zeta \in A$.*

Proof. Apply lemma 2.3 on the dense open set D of conditions which decide the value of ζ . Then there exists $q \geq^* p$ and $\alpha < \kappa$ such that, for every $r \geq q \restriction \alpha$ there exists $r' \geq r$ such that $r' \restriction q \restriction \alpha \Vdash \zeta$. Let $A \in V$ be the set of all possible values of ζ as decided by some extension of q . We argue that $|A| < \kappa$.

Assume that $q' \geq q$ decides the value of ζ . Denote $r = q' \restriction \alpha$. Then there exists $r' \geq r$ such that $r' \restriction q \restriction \alpha \Vdash \zeta$. So both conditions q' , $r' \restriction q \restriction \alpha$ decide the value of ζ ; but those conditions are compatible, since $r' \geq p' \restriction \alpha$, and $q' \restriction \alpha \geq q \restriction \alpha$. This shows that every element of A can be realized as the decided value of ζ by a condition of the form $r \restriction q \restriction \alpha$ for some $r \in P_\alpha$. But the cardinality of the set of such conditions is strictly below κ , since $|P_\alpha| < \kappa$, by the assumption on the cardinality of the forcings Q_β for $\beta < \kappa$.

□

Corollary 2.6 *Let $G_\kappa \subseteq P_\kappa$. Then κ remains inaccessible in $V[G_\kappa]$.*

Proof. We concentrate on the proof that κ remains a regular cardinal after forcing with P_κ , since it's routine to verify that it remains strong limit.

Assume that $\check{f}$ is a P_κ -name for a function from some ordinal $\tau < \kappa$ to κ . Let $p \in P_\kappa$ be a condition which forces this. We argue that there exists $p^* \geq p$ and some $\mu^* < \kappa$, such that $p^* \Vdash \text{rng}(\check{f}) \subseteq \mu^*$.

Let $G_{\tau+1} \subseteq P_{\tau+1}$ be an arbitrary generic extension containing $p \restriction \tau + 1$. We prove that, in $V[G_{\tau+1}]$, there exist $q \geq^* p \restriction \tau + 1$ and $\mu < \kappa$ such that $q \Vdash \text{rng}(\check{f}) \subseteq \mu$. Once we prove that, we are done: let $\check{q}, \check{\mu}$ be $P_{\tau+1}$ -names which are forced by $p \restriction \tau + 1$ to have the above properties. Let $\mu^* < \kappa$ be an upper bound on the set of possible values of $\check{\mu}$, as forced by extensions of $p \restriction \tau + 1$. Since $|P_{\tau+1}| < \kappa$, this set is bounded in κ , and thus there exists such an upper bound below κ . Then $p^* = p \restriction \tau + 1 \restriction \check{q} \Vdash \text{rng}(\check{f}) \subseteq \mu^*$, as desired.

Work in $V[G_{\tau+1}]$. Apply lemma 2.5 over and over to construct a $\leq^*$ -increasing sequence of conditions $\langle p_\xi : \xi \leq \tau \rangle$ in $P_\kappa \setminus \tau + 1$, such that, for each $\xi < \tau$ there exists some $\mu_\xi < \kappa$ such that $p_\xi \Vdash \check{f}(\xi) < \mu_\xi$. Note that in limit steps (including the last step) we may take upper bound, since the direct extension order of $P_\kappa \setminus \tau + 1$ is more than τ -closed. Finally, $q = p_\tau$ forces that the image of $\check{f}$ is bounded by $\mu = \bigcup_{\xi < \tau} \mu_\xi < \kappa$.

□

Theorem 2.7 *Let $S \subseteq \kappa$ be a stationary set consisting of singulars. Let $G_\kappa \subseteq P_\kappa$. Then S remains stationary in $V[G_\kappa]$.*

Proof. Let $C \subseteq \kappa$ be a club in $V[G]$. Let $p \in G$ be condition which forces this.

Work in V . Pick an elementary submodel $M \preceq H_\chi$ such that:

1. $|M| = \delta < \kappa$,
2. $M \cap \kappa = \delta$,
3. $\delta \in S$ (in particular, δ is singular),
4. $\text{cof}(\delta) > M \subseteq M$,
5. $\kappa, P_\kappa, S, \mathcal{C}, p \in M$.

Pick a cofinal sequence in δ , $\langle \delta_i \mid i < \text{cof}(\delta) \rangle$.

Apply lemma 2.5. Construct (in V) a $\leq^*$ -increasing sequence of conditions in the forcing $P_\kappa \setminus \text{cof}(\delta)$, $\langle p_\xi : \xi \leq \text{cof}(\delta) \rangle$, such that each condition p_ξ belongs to M .

We first construct $p_0 \geq^* p$ in M , such that $p_0 \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$, and, using Lemma 2.5,

$$p_0 \restriction \text{cof}(\delta) + 1 \Vdash \exists \alpha < \kappa, p_0 \restriction \text{cof} \delta + 1 \Vdash \min(\mathcal{C} \setminus \delta_0) < \alpha.$$

Let α_0 be the least upper bound of the set of all possible values for the $P_{\text{cof}(\delta)+1}$ -name $\check{\alpha}$. Then $p_0 \Vdash \min(\mathcal{C} \setminus \delta_0) < \alpha_0$. Note that by elementarity, $\alpha_0 < \delta$.

Assuming that $i < \text{cof}(\delta)$ and p_i, α_i have been defined and both are in M , and let us define p_{i+1}, α_{i+1} . Let $p_{i+1} \geq^* p_i$ in M be such that $p_{i+1} \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$, and there exists $\alpha_{i+1} < \kappa$, $p_{i+1} \Vdash \min(\mathcal{C} \setminus \max\{\delta_{i+1}, \alpha_i\}) < \alpha_{i+1}$. Take α_{i+1} to be minimal with this property. Then $\alpha_{i+1} < \delta$.

For the limit step, assume that $j < \text{cof}(\delta)$ is limit and $\langle p_i : i < j \rangle, \langle \alpha_i : i < j \rangle$ have been constructed. Use the fact that M is closed under $< \text{cof}(\delta)$ -sequences to find an upper bound $q \in M$ of $\langle p_i : i < j \rangle$, such that $q \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$. We used here the fact that $P_\kappa \setminus \text{cof}(\delta) + 1$ is $\text{cof}(\delta)$ -closed. Finally, let $p_j \geq^* q$ be chosen in M such that $p_j \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$ and, for some $\alpha_j < \kappa$, $p_j \Vdash \min(\mathcal{C} \setminus \max\{\delta_j, \sup\{\alpha_i : i < j\}\}) < \alpha_j$. We used here again the fact that M is closed under $< \text{cof}(\delta)$ -sequences and thus $\langle \alpha_i : i < j \rangle \in M$. Let α_j be the minimal with the above property. Then $\alpha_j \in M$.

This concludes the inductive construction. In the final limit step, take, in V , an upper bound q^* for all the conditions $\langle p_i : i < \text{cof}(\delta) \rangle$. Then $q^* \Vdash \delta = \bigcup_{i < \text{cof}(\delta)} \alpha_i \in \mathcal{C}$, since C is forced by p to be closed. Thus $\delta \in C \cap S$, as desired.

□

Remark 2.8 Basically the same argument works for κ replaced by κ^+ .

Now let us try to extend the theorem to S 's which consists of regular cardinals. We deal with three supports separately.

First point out the following:

Theorem 2.9 *Suppose that the Easton support is used in P_κ . Assume that $S \subseteq \kappa$ is stationary. Then S remains stationary in $V[G]$.*

Proof. If κ is a Mahlo cardinal, the Easton support iteration is κ -c.c. and thus preserves stationary subsets of κ . Thus, we can assume that κ is not Mahlo. In this case, there exists a club $C \subseteq \kappa$ of singular cardinals. Thus, by shrinking S we can assume that it consists of singulars. Then, Theorem 2.7 applies.

□

Turn to the full support.

Theorem 2.10 *Suppose that P_κ is the full support iteration.*

Assume that for every $\beta < \kappa$, for every $p, q, r \in Q_\alpha$, if $p, q \geq^ r$, then there is $t \in Q_\alpha$ such that $t \geq^* p, q$.*

Let $S \subseteq \kappa$ be a stationary such that, for every $\alpha \in S$, $\langle Q_\alpha, \leq_{Q_\alpha}^ \rangle$ is forced (by the weakest condition of P_α) to be $|\alpha|^+$ -complete.*

Then S remains stationary in $V[G]$.

Proof. Suppose otherwise. Pick some $p \in P_\kappa$ and a name $\mathcal{C}$ such that

$$p \Vdash \mathcal{C} \text{ is a club in } \kappa \text{ and } \mathcal{C} \cap S = \emptyset.$$

Pick now $M \preceq H_\chi$ and $\delta \in S$, as in Theorem 2.7.

It is enough to find a condition $r \geq p$ which forces “ $\mathcal{C}$ is unbounded in δ ” in order to derive a contradiction.

Suppose that there is no such r . Let $G_\delta \subseteq P_\delta$ be a generic with $p \restriction \delta \in G_\delta$. Then, in $V[G_\delta]$,

$$p \restriction \delta \Vdash \mathcal{C}_{G_\delta} \cap \delta \text{ is bounded in } \delta.$$

We have $\delta \in S$, so, by the assumption of the theorem, $\mathcal{C}_\delta$ is δ^+ -complete, and hence, P_κ/G_δ is δ^+ -complete. Then there are $p' \in P_\kappa/G_\delta, p' \geq^* p \restriction \delta$ and $\rho < \delta$ such that

$$p' \Vdash \mathcal{C}_{G_\delta} \cap \delta \subseteq \rho.$$

Pick some $t \in G_\delta, t \geq p \restriction \delta$ such that

$$t \restriction p' \restriction \delta \Vdash \mathcal{C}_{G_\delta} \cap \delta \subseteq \rho.$$

Since $t \geq p \restriction \delta$, there exists $\gamma < \delta$ such that $t = t \restriction \gamma \smallfrown t \restriction \delta \setminus \gamma$ and $t \restriction \delta \setminus \gamma \geq^* p \restriction [\gamma, \delta)$. In particular, $t \restriction \gamma \restriction p' \restriction \delta \geq^* p \restriction \gamma$.

Now work in M above the condition p . Note that γ, ρ above are below δ and thus are in M . Also $p, \mathcal{C} \in M$, and recall that

$$p \Vdash \mathcal{C} \text{ is a club in } \kappa$$

Let $\zeta \in M$ be a P_κ -name such that $p \Vdash \zeta = \min(\mathcal{C} \setminus \check{\rho} + 1)$. By Lemma 2.5, we can find (in M) a P_γ -name for a condition $q_1 \geq^* p \restriction \gamma$ such that $p \restriction \gamma \restriction q_1 \restriction \delta \Vdash \exists \mu < \kappa \ q_1 \restriction \delta \restriction \zeta < \mu$. Let $\mu \in M$ be a P_γ -name such that $p \restriction \gamma \restriction q_1 \restriction \delta \restriction \zeta < \mu$. In M , let μ^* be the supremum of all possible values of μ , as forced by extensions of $p \restriction \gamma$. Then $\mu^* < \delta$ (since $\mu^* \in M \cap \kappa$). By elementarity, $p \restriction \gamma \restriction q_1 \restriction \delta \restriction \zeta < \mu^*$ holds in V as well.

Finally, let $p^* \in P_\kappa$ be a condition such that $q^* \restriction \gamma = t \restriction \gamma$ and $p^* \restriction \gamma$ direct extends both the conditions $t \restriction \gamma \restriction p'$, q_1 . Note that, since $t \restriction \gamma \restriction p'$ and q_1 direct extend $p \restriction \gamma$, the extra assumption of the theorem allows us to construct such p^* .

Then on the one hand, p^* forces that $\mathcal{C} \cap \delta \subseteq \rho$; on the other hand, it forces that $\min(\mathcal{C} \setminus \rho + 1) \leq \mu^* < \delta$. A contradiction.

□

Turn now to the non-stationary support.

Theorem 2.11 *Assume that the non-stationary support is used. Let $S \subseteq \kappa$ be stationary. Then S is stationary in $V[G_\kappa]$.*

Proof. Let $\mathcal{C}$ be a P_κ -name and $p \in P_\kappa$, $p \Vdash \mathcal{C}$ is a club disjoint to S .

We construct:

1. $\leq^*$ -increasing sequence of conditions $\langle p_i : i < \kappa \rangle$,
2. decreasing sequence of clubs $\langle C_i : i < \kappa \rangle$, each is disjoint to the support of p_i ,
3. increasing continuous sequences of ordinals $\langle \nu_i : i < \kappa \rangle$, $\langle \alpha_i : i < \kappa \rangle$.

The sequence of conditions will be a fusion sequence, in the sense that, for every $i < j$, $p_i \leq^* p_j$ and $p_j \restriction \nu_i = p_i \restriction \nu_i$.

Let $p_0 = p$ and $\alpha_0 = 0$. Pick a club C_0 disjoint to $\text{supp}(p_0)$. Let $\nu_0 = \min(C_0)$. Let $G_{\nu_0} \subseteq P_{\nu_0}$ be a generic with $p_0 \restriction \nu_0 \in G_{\nu_0}$. Apply Lemma 2.5, inside $V[G_{\nu_0}]$, and find $p'_1 \in P_\kappa/G_{\nu_0}$, $p'_1 \geq^* p_0 \restriction \nu_0$ and $A'_1 \subseteq \kappa$, $|A'_1| < \kappa$ such that $p'_1 \Vdash \min(\mathcal{C}) \in A'_1$. Now back to V , we have $|P_{\nu_0}| < \kappa$, hence there is A_1 , $|A_1| < \kappa$ such that

$$p_1 = p_0 \restriction \nu_0 \widehat{\smile} p'_1 \Vdash \min(\mathcal{C}) \in A_1.$$

Set $\alpha_1 = \sup(A_1)$.

Next, we pick a club $C_1 \subseteq C_0$ disjoint to $\text{supp}(p_1)$. Let $\nu_1 = \min(C_1 \setminus \nu_0 + 1)$. Let $G_{\nu_1} \subseteq P_{\nu_1}$ be a generic with $p_1 \restriction \nu_1 \in G_{\nu_1}$. Apply Lemma 2.5, inside $V[G_{\nu_1}]$, and find $p'_2 \in P_\kappa/G_{\nu_1}$, $p'_2 \geq^* p_1 \restriction \nu_1$ and $A'_2 \subseteq \kappa$, $|A'_2| < \kappa$ such that $p'_2 \Vdash \min(\mathcal{C} \setminus \alpha_1 + 1) \in A'_2$. Now back to V , we have $|P_{\nu_1}| < \kappa$, hence there is A_2 , $|A_2| < \kappa$ such that

$$p_2 = p_1 \restriction \nu_1 \widehat{\smile} p'_2 \Vdash \min(\mathcal{C} \setminus \alpha_1 + 1) \in A_2.$$

Set $\alpha_2 = \sup(A_2)$.

We do the same at each successor stage $i < \kappa$ of the construction.

Suppose now that $i < \kappa$ is limit. Set $\alpha_i = \bigcup_{j < i} \alpha_j$, $\nu_i = \bigcup_{j < i} \nu_j$ and let p_i be the coordinateswise union of $\langle p_j : j < i \rangle$. Note that the facts that $\langle p_j : j < i \rangle$ is a fusion sequence, $\nu_i \notin C_j$ for all $j < i$, and $P_\kappa \setminus G_{\nu_{i+1}}$ is more than $|i|$ -closed, ensure that $p_i \in P_\kappa$ is a legitimate condition. Let $C_i = \bigcap_{j < i} C_j$. Pick $\nu_{i+1} = \min(C_i) \setminus \nu_i + 1$. Continue as above.

Finally, let p_κ be the coordinateswise union of $\langle p_i : i < \kappa \rangle$. It is in P_κ since for every $i < j$, $p_i \leq^* p_j$ and $p_j \restriction \nu_i = p_i \restriction \nu_i$ and $\Delta_{i < \kappa} C_i$ is disjoint to its support.

Now,

$$p_\kappa \Vdash \{\alpha_i \mid i \text{ is limit}\} \subseteq \mathcal{C}.$$

But $\{\alpha_i \mid i \text{ is limit}\}$ is a club in V , and so, $S \cap \{\alpha_i \mid i \text{ is limit}\} \neq \emptyset$. Contradiction.

□

Remark 2.12 A similar argument can be used to show that non-stationary support iterations preserve stationary subsets of κ^+ .⁶ The main difference is the use of elementary substructures. We sketch the argument below.

Let $S \subseteq \kappa^+$ be stationary. Assume that $p \in P_\kappa$ forces that $\mathcal{C}$ is a club in κ^+ disjoint from S . Pick an elementary substructure $M \preceq H_\chi$ (for χ high enough), such that $|M| = \kappa$, $\gamma := \sup(M \cap \kappa^+) \in S$, M is closed under less-than $\text{cf}(\gamma)$ -sequences of its elements, and M contains all the relevant parameters as elements (namely, $\kappa, P_\kappa, S, p, \mathcal{C} \in M$). We assume $\text{cf}(\gamma) = \kappa$ and fix $\langle \gamma_i : i < \kappa \rangle$ cofinal in γ (the case where $\text{cf}(\gamma) < \kappa$ is similar, even simpler). Construct increasing sequences $\langle p_i : i < \kappa \rangle, \langle C_i : i < \kappa \rangle, \langle \nu_i : i < \kappa \rangle$ as before, and an increasing, continuous sequence $\langle \alpha_i : i < \kappa \rangle$ cofinal in γ , that dominates $\langle \gamma_i : i < \kappa \rangle$. The construction is internal to M , in the sense that each strict initial segment of the above sequences belongs to M (but the entire sequences are external to M). The construction is identical to the one from the proof of Theorem 2.11, with the minor adaption that for each $i < \kappa$, $\gamma_i \leq \alpha_i$ and

$$p_{i+1} \Vdash \min(\mathcal{C} \setminus \alpha_i + 1) \leq \alpha_{i+1}.$$

At the final limit step, let p_κ be the coordinatewise union of $\langle p_i : i < \kappa \rangle$. Then $p_\kappa \Vdash \gamma \in \mathcal{C} \cap S$, which is a contradiction.

We proceed and discuss preservation of cardinals after forcing with P_κ with various supports. In the Easton support case, $|P_\kappa| = \kappa$, and so κ^+ is preserved. By Corollary 2.6 in [1], κ^+ is preserved after non-stationary support iterations. Let us consider full support iterations. Note that the Magidor iteration of Prikry forcings (from [11]) satisfies κ^+ -c.c.. This could be generalized to a Magidor iteration of arbitrary Prikry-type forcings:

Proposition 2.13 *Suppose that for every $\beta < \kappa$, for every $s, t, r \in \mathcal{Q}_\beta$, if $s \leq_{\mathcal{Q}_\beta}^* t, r$, then there is $e \in \mathcal{Q}_\beta, e \geq_{\mathcal{Q}_\beta}^* t, r$. Then P_κ satisfies κ^+ -c.c.*

In general it turns out that κ^+ may be collapsed with full support iteration.

Proposition 2.14 *Suppose that κ is a measurable cardinal. Let P_κ be the full support iteration of $\text{Col}(\alpha, \alpha^+) = \{f \mid f \in {}^\xi \alpha^+, \xi < \alpha\}$, for every regular $\alpha < \kappa$. Then κ^+ is collapsed in V^{P_κ} .*

Proof. Let U be a normal measure over κ .

We start with the following claim:

⁶The fact that κ^+ itself is preserved as a cardinal is also required here, and it appears in [1], Corollary 2.6; we will discuss preservation of κ^+ in more detail in the rest of the subsection.

Claim 1 Let $p = \langle \underset{\sim}{p}(\beta) \mid \beta < \kappa \rangle \in P_\kappa$. Then there are $A^* \in U$, $\tau^* < \kappa$ and $p^* \geq p$ such that $p^* \restriction \beta \Vdash \text{dom}(\underset{\sim}{p}^*(\beta)) = \tau^*$, for every $\beta \in A^*$.

Proof. For every regular $\beta < \kappa$ there are $s_\beta \in P_\beta$, $s_\beta \geq p \restriction \beta$ and $\tau_\beta^0 < \beta$ such that $s_\beta \Vdash \text{dom}(\underset{\sim}{p}(\beta)) = \tau_\beta^0$.

Find $A'_0 \in U$ and $p_0 \in P_\kappa$ such that $p_0 \restriction \beta = s_\beta$, for every $\beta \in A'_0$. For example, take $p_0 = [\beta \mapsto s_\beta]_U$.

Then we consider a regressive function $\beta \mapsto \tau_\beta^0$ on A'_0 . Find $A_0 \subseteq A'_0$, $A_0 \in U$ and $\tau^0 < \kappa$ such that $\tau_\beta^0 = \tau^0$, for every $\beta \in A_0$.

Repeat the process with p_0 replacing p and find $A_1 \subseteq A_0$, $A_1 \in U$, p_1 and τ^1 such that $p_1 \restriction \beta \Vdash \text{dom}(\underset{\sim}{p}_0(\beta)) = \tau^1$.

Continue by induction. Let $A^* = \bigcap_{n < \omega} A_n$ and p^* be the coordinatewise union of p_n 's. Set $\tau^* = \bigcup_{n < \omega} \tau^n$.

Then $p^* \restriction \beta \Vdash \text{dom}(\underset{\sim}{p}^*(\beta)) = \tau^*$, for every $\beta \in A^*$, will be as desired.

□ of the claim.

For every $\tau < \kappa$, define a maximal antichain A_τ in P_κ . Proceed as follows.

Let us pick functions $\langle h_\gamma \mid \gamma < \kappa^+ \rangle$ such that $\text{dom}(h_\gamma) \in U$, for every $\alpha \in \text{dom}(h_\gamma)$, $h_\gamma(\alpha) < \alpha^+$ and $[h_\gamma]_U = \gamma$, for example κ^+ -canonical functions will do the job.

Fix $\tau < \kappa$. Let A_τ be a maximal antichain in P_κ of cardinality κ^+ which consists of $p \in P_\kappa$ such that:

(*) if for some $B \in U$ and $\gamma < \kappa^+$, the condition $t_{B\gamma} = \langle t_{B\gamma}(\alpha) \mid \alpha < \kappa \rangle$ is compatible with p , then, for some $B' \in U$, $p \geq t_{B'\gamma}$,

where, for $E \in U$, $t_{E\gamma}(\alpha) = 0_\alpha$, unless $\alpha \in E \cap \text{dom}(h_\gamma) \setminus \tau + 1$, and if $\alpha \in E \cap \text{dom}(h_\gamma) \setminus \tau + 1$, then $t_{E\gamma}(\alpha) = \{(\tau, h_\gamma(\alpha))\}$, i.e. the value of the generic function for α at τ is $h_\gamma(\alpha)$.

Let $\langle p_i^\tau \mid i < \kappa^+ \rangle$ be an enumeration of A_τ .

Let $G \subseteq P_\kappa$ be a generic. Define $F : \kappa \rightarrow (\kappa^+)^V$ by setting $F(\tau) = i$ iff $p_i^\tau \in G$.

We claim that such F is onto.

Suppose otherwise. Pick some $p \in G$ and $\eta < \kappa^+$ such that $p \Vdash \text{rng}(\underset{\sim}{F}) \subseteq \eta$.

Apply Claim 1. Let $A^* \in U$, $\tau^* < \kappa$ and $p^* \geq p$ be as in the conclusion of the claim.

Now for every $\gamma < \kappa^+$, we can extend p^* to a condition p^γ by adding a pair $(\tau^*, h_\gamma(\alpha))$ to $p^*(\alpha)$, for every $\alpha \in A^* \cap \text{dom}(h_\gamma) \setminus \gamma + 1$. Note that if $\gamma \neq \gamma'$, then $p^\gamma, p^{\gamma'}$ are incompatible. So, the set $\{p^\gamma \mid \gamma < \kappa^+\}$ consists of κ^+ -many incompatible conditions.

Then, each of p^γ 's must be compatible with a member of A_{τ^*} with index below η . Hence, there is $i^* < \eta$ such that $p_{i^*}^{\tau^*}$ is compatible with κ^+ -many p^γ 's. Pick two of them $\gamma \neq \gamma'$.

By (*), then $p_{i^*}^{\tau^*} \geq t_{B\gamma}, t_{B'\gamma'}$, for some $B, B' \in U$. However, $\gamma \neq \gamma'$ implies that there is

$\alpha \in B \cap B'$, such that $h_\gamma(\alpha) \neq h_{\gamma'}(\alpha)$, and this is impossible due to the compatibility. Contradiction.

□.

We conjecture that the measurability assumption can be much weakened. However, the following positive result can be proved:

Proposition 2.15 *Suppose that there is a club $C \subseteq \kappa$ such that for every $\alpha \in C$, $\langle \mathcal{Q}_\alpha, \leq_{\mathcal{Q}_\alpha}^* \rangle$ is forced to be α^+ -closed. Then P_κ preserves κ^+ .*

In particular, if κ is not a Mahlo cardinal, then P_κ preserves κ^+ .

Proof. Let $p \in P_\kappa$ and f be a name such that

$$p \Vdash f : \kappa \rightarrow \kappa^+.$$

Fix a club C such that for every $\alpha \in C$, $\langle \mathcal{Q}_\alpha, \leq_{\mathcal{Q}_\alpha}^* \rangle$ is forced to be α^+ -closed. Assume also that for every $\alpha \in C$, for every $\beta < \alpha$, $|P_\beta| < \alpha$. Let $\langle \alpha_i \mid i < \kappa \rangle$ be an increasing continuous enumeration of C . Apply Lemma 2.5 and find $p_0 \geq^* p$, $p_0 \restriction \alpha_0 = p \restriction \alpha_0$ and $\eta_0 < \kappa^+$ such that $p_0 \Vdash f(0) < \eta_0$.

Continue by induction and define a $\leq^*$ -increasing sequence $\langle p_i \mid i < \kappa \rangle$ and sequence $\langle \eta_i \mid i < \kappa \rangle$ of ordinals below κ^+ such that

1. $p_i \Vdash f(i) < \eta_i$,
2. for every $i < j$, $p_j \restriction \alpha_i = p_i \restriction \alpha_i$,

There is no problem at limit stages i , since $\langle P_\kappa \setminus \alpha_i, \leq^* \rangle$ is α_i^+ -closed since $\alpha_i \in C$.

The second item insures that there is $p^* \in P_\kappa$ such that $p^* \geq^* p_i$, for every $i < \kappa$. Then

$$p^* \Vdash \text{rng}(f) \subseteq \bigcup_{i < \kappa} \eta_i.$$

□

3 Non-stationary support iterations

We assume GCH throughout as before. Let κ be an inaccessible cardinal.

Let $\langle P_\alpha, \mathcal{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a non-stationary support iteration of Prikry-type forcings, with the properties stated in the introduction.

Let I be a stationary subset of κ which consists of singular cardinals μ such that for every $\gamma < \mu$, $|P_\gamma| < \mu$.

Assume that-

If $\alpha \in I$, then $\Vdash_{P_\alpha} \langle P_\kappa \setminus \alpha, \leq^* \rangle$ is α^{++} -closed.

Note that for a singular cardinal α , we have $\Vdash_{P_\alpha} \langle P_\kappa \setminus \alpha, \leq^* \rangle$ is α^+ -closed. We will need a slightly more closure. A typical situation is where κ is Mahlo, Q_α is trivial at every accessible α and each forcing Q_α has cardinality below the least inaccessible above α .

Theorem 3.1 P_κ does not add fresh unbounded subsets to κ .

Proof. Recall the following fusion lemma for the non-stationary support iteration of Prikry forcings⁷:

Lemma 3.2 Let $p \in P_\kappa$. For every $\beta < \kappa$, let $F(\beta)$ be a P_β -name for a $\leq^*$ -dense open subset of $P \setminus \beta$ above $p \restriction \beta$, and assume that this is forced by $p \restriction \beta$. Then there exist $p^* \geq^* p$ and a club $C \subseteq \kappa$ such that for every singular $\beta \in C$, $p^* \restriction \beta \Vdash p^* \setminus \beta \in F(\beta)$.

Let $G \subseteq P_\kappa$ be generic over V , and assume for contradiction that there exists a function $f \in 2^\kappa$ which is the characteristic function of a fresh subset of κ . Let $\check{f}$ be a P_κ -name for it, and assume that this is forced by some condition in G . For simplicity, assume that this is the weakest condition.

Let $\zeta \in \kappa \cap I$ be the least ordinal for which a new subset is added in the extension from V to $V[G]$. Such ζ exists, since the forcings Q_α (for $\alpha < \kappa$) have cardinality below κ , and at least one of them is non-trivial.

Note that $\zeta \in \kappa \cap I$, $\langle P_\kappa \setminus \zeta, \leq^* \rangle$ is ζ^{++} -closed. Pick a condition $q \in P_\zeta$ which forces that a new subset is added to ζ . For simplicity, assume that the weakest condition in P_ζ forces this (else, work above a condition in P_κ whose restriction to P_ζ equals q).

We divide into two cases:

Case 1.⁸ There exists $\mu \in (\zeta, \kappa) \cap I$ and a condition $p^* \in P_\mu$ which forces that the following property holds:

$$\begin{aligned} &\exists p \in G_\mu \exists s \in P_\kappa \setminus \mu \forall r \geq^* s \exists \xi < \kappa \exists r_0, r_1 \geq^* r, \\ &V \models (p \restriction r_0 \parallel \check{f} \restriction \xi, p \restriction r_1 \parallel \check{f} \restriction \xi), \text{ and the decisions are different.} \end{aligned}$$

⁷The fusion property for non-stationary support iterations is due to Friedman and Magidor [2]. A version suitable for iterations of Prikry-type forcings appeared in [1]. The proof is basically given in lemma 3.3 in [4].

⁸It basically repeats those of 4.11, [4].

(here, G_μ denotes the canonical name for the generic set for P_μ). By extending p^* , we can decide the value of p in the statement above, and thus assume that $p^* \geq p$. Let $\underline{s}$ be a P_μ -name for s from the above property, and assume that this is forced by p^* .

Let us apply the same methods as in the main lemma in [7]. We construct, in V , a binary tree of conditions, $\langle \langle p^*, \underline{s}_\sigma \rangle : \sigma \in {}^\mu 2 \rangle$ and a tree of functions $\langle b_\sigma : \sigma \in {}^\mu 2 \rangle$ such that $\underline{s}_\emptyset = \underline{s}$, and for every $\sigma \in {}^\mu 2$:

1. $\forall i < 2, \langle p^*, \underline{s}_{\sigma \smallfrown \langle i \rangle} \rangle \parallel \underline{f} \restriction lh(b_{\sigma \smallfrown \langle i \rangle}) = b_{\sigma \smallfrown \langle i \rangle}$.
2. $b_{\sigma \smallfrown \langle 0 \rangle} \perp b_{\sigma \smallfrown \langle 1 \rangle}$.
3. $\forall i < 2, p^* \Vdash \underline{s}_{\sigma \smallfrown \langle i \rangle} \geq^* s_\sigma$.
4. If $lh(\sigma)$ is limit, then p^* forces that $\underline{s}_\sigma$ is an upper bound, with respect to the direct extension order, of $\langle \underline{s}_{\sigma \restriction \xi} : \xi < lh(\sigma) \rangle$.
5. b_σ is an end extension of $b_{\sigma \restriction \xi}$ for every $\xi < lh(\sigma)$.

Now assume that $g \subseteq P_\mu$ is generic over V with $p^* \in g$. In $V[g]$, let $h \in 2^{<\mu}$ be the characteristic function of a new subset of μ (such a new subset exists because μ is above ζ). h defines a branch through the binary tree, $\langle \langle p^*, \underline{s}_{h \restriction \xi} \rangle : \xi < \mu \rangle$. Since $\langle \underline{s}_{h \restriction \xi} : \xi < \mu \rangle$ form a $\leq^*$ -increasing sequence, there exists an upper bound $s^* \in P \setminus \mu$, which extends all the conditions in the sequence. Thus, there exists an upper bound for the branch, of the form $\langle p^*, \underline{s}^* \rangle$. It forces that–

$$b = \bigcup_{\xi < \mu} b_{h \restriction \xi}$$

is an initial segment of $\underline{f}$. We argue that this must be a strict initial segment of $\underline{f}$. Indeed, otherwise, $(\text{cof}(\kappa))^{V[G_\mu]} \leq \mu$. But, since κ is inaccessible, $G_\mu \subseteq P_\mu$ is a forcing whose cardinality is strictly below κ , so it preserves cofinalities greater or equal to κ .

Therefore, b is a strict initial segment of $\underline{f}$, and thus $b \in V$. So h can be defined, in V , using the binary tree and the set b . This is a contradiction to the choice of h .

Case 2. For every $\mu \in (\zeta, \kappa) \cap I$, every condition in P_μ forces that–

$$\forall p \in G_\mu \forall s \in P_\kappa \setminus \mu \exists r \geq^* s \forall \xi < \kappa \forall r_0, r_1 \geq^* r,$$

$$V \models \text{If } p \smallfrown r_0 \parallel \underline{f} \restriction \xi \text{ and } p \smallfrown r_1 \parallel \underline{f} \restriction \xi \text{ then the decisions are the same.}$$

Define for every $\mu < \kappa$

$$e(\mu) = \{r \in P_\kappa \setminus \mu : \forall p \in G_\mu \forall \xi < \kappa \forall r_0, r_1 \geq^* r,$$

$$V \models \text{If } p \smallfrown r_0 \parallel \underline{f} \restriction \xi \text{ and } p \smallfrown r_1 \parallel \underline{f} \restriction \xi \text{ then the decisions are the same.}\}$$

Claim 2 Suppose that $\mu \in (\zeta, \kappa) \cap I$. Then $e(\mu)$ is a dense open in $\langle P_\kappa \setminus \mu, \le^* \rangle$.⁹

Proof. Just note that $|G_\mu| \leq \mu^+$ and $\langle P_\kappa \setminus \mu, \le^* \rangle$ is μ^{++} -closed.

□ of the claim.

Given $\mu \in (\zeta, \kappa) \cap I$ as above, the following set is also forced to be $\le^*$ -dense open in $P \setminus \mu$:

$$d(\mu) = \{r \in P \setminus \mu : \exists g \in 2^\mu, r \Vdash \check{f} \restriction \mu = g\}.$$

The $\le^*$ -density of $d(\mu) \subseteq P \setminus \mu$ follows as well from the fact that the direct extension order of $P \setminus \mu$ is more than μ -closed.

We can now apply the standard fusion argument 3.2. There exists $p \in P_\kappa$ and a club $C \subseteq \kappa$ such that $\min(C) > \zeta$, and, for every $\mu \in C \cap I$,

$$p \restriction \mu \Vdash p \setminus \mu \in d(\mu) \cap e(\mu).$$

For each $\mu \in C \cap I$, there exists a condition in G of the form $q_\mu \frown p \setminus \mu$, where $q_\mu \in P_\mu$, which decides the value of $\check{f} \restriction \mu$. The reason is that $p \restriction \mu$ forces that $p \setminus \mu$ is in $d(\mu)$, and thus the value of $\check{f} \restriction \mu$ is decided by the forcing P_μ .

For each such μ , q_μ is an extension of $p \restriction \mu$, and thus there exists a finite set $b_\mu \subseteq \mu$ such that at every $\delta \in \mu \setminus b_\mu$, $q_\mu(\delta)$ direct extends $p(\delta)$ (as forced by $q_\mu \restriction \delta$).

The function $\mu \mapsto \max b_\mu$ is a regressive function in $V[G]$, and its domain is the set $C \cap I$. $C \cap I$ is stationary in V since I is assumed to be stationary. By theorem 2.2, it is also stationary in $V[G]$. Since κ is still regular in $V[G]$, we can find an unbounded subset $S \subseteq \kappa$ and an ordinal $\mu^* < \kappa$ such that for every $\mu \in S$, $b_\mu \subseteq \mu^*$. By increasing μ^* , we can assume that it belongs to $C \cap I$.

Now, shrink S further to stabilize the function $\mu \mapsto q_\mu \restriction \mu^*$. This is possible since S is unbounded in κ , and $q_\mu \restriction \mu^*$ is a condition in P_{μ^*} which has a small cardinality (and, again, κ is inaccessible in $V[G]$).

So we can assume that there exists a condition $q^* \in P_{\mu^*}$, such that for every $\mu \in S$, there exists some direct extension $r_\mu \in P \setminus \mu^*$ of $p \setminus \mu^*$, such that—

$$q^* \frown r_\mu \Vdash \check{f} \restriction \mu$$

and $q^* \frown r_\mu \in G$.

⁹It is the only place in the proof where μ^{++} -completeness is used.

Recall that $\mu^* \in C \cap I$ and the condition p obtained by fusion above also satisfies that $p \restriction \mu^* \Vdash p \restriction \mu^* \in e(\mu^*)$. Since $q^* \geq p \restriction \mu^*$, the condition q^* forces that, for every $\xi < \kappa$, any pair of direct extensions of $p \restriction \mu^*$ which decide $\check{f} \restriction \xi$, decide this initial segment the same way.

It follows that $q^* \frown p \restriction \mu^*$ decides $\check{f}$ entirely, and forces it to be the following function of V :

$$h = \bigcup_{\mu \in \kappa \setminus \mu^*} \{g \in 2^\mu : \text{there exists a } P_{\mu^*}\text{-name for an extension } \check{r} \geq^* p \restriction \mu^*, \\ \text{such that } q^* \frown r \Vdash \check{f} \restriction \mu = g\}$$

which is a contradiction. $\square$

Remark 3.3 Lemma 3.2 implies that $V[G] \models 2^\kappa = \kappa^+$. Indeed, assume that A is a subset of κ in $V[G]$. Let $\check{A}$ be a P_κ -name for it. For every singular $\beta < \kappa$, define in V^{P_β} the set–

$$F(\beta) = \{q \in P_\kappa \restriction \beta : \exists A_\beta \subseteq \beta, q \Vdash \check{A} \cap \beta = A_\beta\}.$$

Note that β is singular and thus $F(\beta)$ is $\leq^*$ -dense open. Let $p^* \in G$ and $C \subseteq \kappa$ be such that for every singular $\beta \in C$, $p^* \restriction \beta \Vdash p^* \restriction \beta \in F(\beta)$. Then there exists a P_β -name $\check{A}_\beta$ for a subset of β , such that $p^* \restriction \beta \Vdash \check{A} \cap \beta = (\check{A}_\beta)_{\mathcal{G} \restriction P_\beta}$. Then $A = (\check{A})_G$ can be computed in $V[G]$ from the sequence $\langle \check{A}_\beta : \beta \in C \rangle$. By using canonical names for bounded subsets of κ , and by GCH in V , there are at most κ^+ such sequences. So there are at most κ^+ –many subset of κ in $V[G]$.

The situation with higher cardinals was clarified in [4]. The following was shown basically in [4], 4.11:

Lemma 3.4 P_κ does not add fresh unbounded subsets to κ^+ , or to any cardinal λ of V with $\text{cof}(\lambda) > \kappa$.

Proof. The proof is a variation of the proof of theorem 3.1, and it basically appears in [4]. Assume that $\check{f}$ is a P_κ -name for the characteristic function of a fresh unbounded subset of λ . Divide into cases as in the proof of theorem 3.1. Case 1 remains the same. Case 2 is simplified, since the sets $d(\mu)$ is no longer required. Indeed, in the notations of the proof of

theorem 3.1, assume that $p \in P_\kappa$ is a condition and $C \subseteq \kappa$ is a club, such that for every $\mu \in C \cap I$,

$$p \restriction \mu \Vdash p \restriction \mu \in e(\mu)$$

Now, work in $V[G]$. For each $\xi < \lambda$, let $q_\xi \in P_\kappa$ be an extension of p which decides $\tilde{f} \restriction \xi$. Let $b_\xi \subseteq \kappa$ be a finite set such that, for every $\alpha \in b_\xi$, $q_\xi \restriction \alpha \Vdash q_\xi(\alpha) \geq^* p(\alpha)$. Let $\mu_\xi < \kappa$ be an upper bound on b_ξ . Since $\text{cof}(\lambda) > \kappa$ in V , the same holds true in $V[G]$ as well (by the same proof as in corollary 2.6). Thus, there exists $\mu^* \in C \cap I$ such that, for an unbounded $S \subseteq \lambda$, $\mu_\xi < \mu^*$. By shrinking S , we can assume that, for some $q^* \in P_{\mu^*}$, $q_\xi \restriction \mu^* = q^*$. Then q^* satisfies that, for every $\xi \in S$, there exists a direct extension $r_\xi \in P \restriction \mu^*$ of $p \restriction \mu^*$, such that $q^* \restriction r_\xi \parallel \tilde{f} \restriction \xi$. Now, as in the proof of case 2 in theorem 3.1, the condition $q^* \restriction p \restriction \mu^*$ forces $\tilde{f}$ to be the following function of V :

$$h = \bigcup_{\xi < \kappa^+} \{g \in 2^\mu : \text{there exists a } P_{\mu^*}\text{-name for an extension } \tilde{r} \geq^* p \restriction \mu^*, \\ \text{such that } q^* \restriction \tilde{r} \Vdash \tilde{f} \restriction \xi = g\}$$

which is a contradiction.

□

We finish with an application for iterations of Prikry-type forcings with the nonstationary support. The referee of [10] asked if such an iteration, below a cardinal κ , can add new measurable cardinals below κ . We show that the answer is negative¹⁰.

Theorem 3.5 *Assume GCH and let κ be an inaccessible cardinal.*

Let P_κ be a non-stationary support iteration of Prikry-type forcings satisfying the conditions from the beginning of the section. Assume:

() For every Mahlo cardinal $\alpha < \kappa$ which is not measurable in V , $\langle P_\kappa \restriction \alpha, \leq^* \rangle$ is α^+ -closed.*

Let λ be a cardinal such that $\forall \tau < \lambda (|P_\tau| < \lambda)$. Let $G \subseteq P_\kappa$ be generic over V . Then If λ is measurable in $V[G]$, it was already measurable in V .

Furthermore, if the assumption () is strengthened to–*

*(**) For every Mahlo cardinal $\alpha < \kappa$, $\langle P_\kappa \restriction \alpha, \leq^* \rangle$ is α^+ -closed.*

then λ is measurable in $V[G]$ if and only if it is measurable in V .

¹⁰The answer for the same question in the full or Easton support is known to be negative, see [3].

Proof. Assume first that $(**)$ holds and λ is a measurable cardinal in V . If $\lambda > \kappa$, the Levy-Solovay Theorem [12] shows that λ is measurable in $V[G]$. If $\lambda \leq \kappa$, standard arguments show that λ remains measurable in $V[G \restriction P_\lambda]$ (see [5]). If $\lambda < \kappa$, the forcing $\langle P_\kappa \setminus \lambda, \leq^* \rangle$ is λ^+ -closed, so $P_\kappa \setminus \lambda$ does not add subsets to λ and λ remains measurable in $V[G]$. Thus, let us concentrate on the other direction assuming $(*)$.

We first recall Lemma 2.1 in [4]: given a forcing notion which does not add new fresh unbounded subsets to cardinals of V in the interval $[\kappa, (2^\kappa)^V] = [\kappa, \kappa^+]$, every κ -complete ultrafilter in the generic extension extends a κ -complete ultrafilter from V . Assume now that α is measurable in $V[G]$, and let $W \in V[G]$ be a nontrivial κ -complete ultrafilter on α . If $\alpha \geq \kappa$, then by the results in this section, P_κ does not add fresh unbounded subsets to α , α^+ , and thus $W \cap V \in V$ by Lemma 2.1 in [4]. Thus, assume that $\alpha < \kappa$, and assume that α is not measurable in V . So, α is a Mahlo cardinal in V . If the forcing $\langle P_\kappa \setminus \alpha, \leq^* \rangle$ is α^{++} -closed, then $W \in V[G \restriction P_\alpha]$.

However we assumed only that $\langle P_\kappa \setminus \alpha, \leq^* \rangle$ is α^+ -closed. In this case W need not be in $V[G \restriction P_\alpha]$.

Proceed then as follows. Work in $V[G \restriction P_\alpha]$. Let $\langle A_i \mid i < \alpha^+ \rangle$ be an enumeration of all subsets of α (such enumeration exists by applying remark 3.3 on P_α). Define a $\leq^*$ -increasing sequence of conditions $\langle p_i \mid i < \alpha \rangle$ in $V[G \restriction P_\alpha]$ such that for every $i < \alpha$, $p_i \Vdash A_i \in \tilde{W}$. Set $W' = \{A_i \mid i < \alpha, p_i \Vdash A_i \in \tilde{W}\}$. Then W' will be an α -complete ultrafilter over α in $V[G \restriction P_\alpha]$.

Apply now 3.1 and 3.4 to P_α . It follows that no fresh subsets are added to α, α^+ . Now, by Lemma 2.1 from [4], $W' \cap V \in V$ is a nontrivial α -complete ultrafilter over α in V . A contradiction.

□

Remark 3.6 1. The closure assumptions made on $\langle P_\kappa \setminus \alpha, \leq^* \rangle$ are needed.

For example, start with $V = L[U]$, where U is a normal ultrafilter over α . Iterate $Cohen(\beta), \beta < \alpha$. Let V be this model. Then α is not a measurable in V . Force with $Cohen(\alpha)$ over V . Then α will be a measurable in the extension. Here we take Q_β to be trivial for every $\beta < \alpha$ and $Q_\alpha = Cohen(\alpha)$.

2. Also, the assumption $\forall \tau < \lambda (|P_\tau| < \lambda)$ is necessary. Just use the previous example (with $\lambda = \alpha$). $Q_0 = Cohen(\alpha)$ resurrects measurability of α .
3. Note that a measurable cardinal in V need not be such in $V[G]$ without assuming $(*)$. Just use the Prikry forcing or the iteration of such forcings.

4 On fresh sets in the Easton support iterations

We start with an easier case of the Easton support iterations with κ being a Mahlo cardinal. κ -c.c. of the forcing will be used to show the following:

Theorem 4.1 *Suppose that κ is a Mahlo cardinal and P_κ is an Easton support iteration. Then no fresh sets are added to κ .*

Proof. Suppose otherwise. Work in V . Let $\dot{A}$ be a name of such subset.

Define a tree T of possibilities as follows. Fix an increasing enumeration $\langle \kappa_\xi \mid \xi < \kappa \rangle$ of all inaccessible cardinals below κ .

For every $\xi < \kappa$, let

$$\text{Lev}_\xi(T) = \{x \subseteq \kappa_\xi \mid \exists p \in P_\kappa \quad p \Vdash \dot{A} \cap \kappa_\xi = x\}.$$

Let $x \in \text{Lev}_\alpha(T), y \in \text{Lev}_\beta(T)$. Set $x >_T y$ iff $\alpha > \beta$ and $x \cap \kappa_\beta = y$.

Then $\langle T, <_T \rangle$ is a κ -tree, since κ is an inaccessible.

Lemma 4.2 *$\langle T, <_T \rangle$ has a κ -branch.*

Proof. Let $\langle x_\gamma \mid \gamma < \kappa \rangle$ be an enumeration of T .

There is a club $C \subseteq \kappa$ such that for every $\gamma, \delta \in C$ the following hold:

1. $\kappa_\gamma = \gamma$,
2. the level of $x_\gamma \geq \gamma$,
3. if $\gamma < \delta$, then the level of $x_\gamma < \delta$.

For every $\gamma \in C$, pick $p_\gamma \in P_\kappa$ such that $p_\gamma \Vdash \dot{A} \cap \kappa_{\xi_\gamma} = x_\gamma$, where $\xi_\gamma \geq \gamma$ denotes the level of x_γ .

Now, κ is a Mahlo cardinal and an Easton support was used, hence there is a stationary $S \subseteq C$ such that for every $\gamma, \delta \in S$, p_γ and p_δ are compatible.

Take any two $\gamma < \delta$ in S . Then $x_\delta \cap \kappa_\gamma = x_\gamma$ due to the compatibility of p_γ and p_δ .

So, $\{x_\gamma \mid \gamma \in S\}$ is a κ -branch.

□

Let now $b = \{x_i \mid i < \kappa\}$ be a maximal κ -branch in T . For every $i < \kappa$ fix $p_i \in P_\kappa$ which witnesses that $x_i \in T$.

By the assumption made, $\bigcup_{i < \kappa} x_i \neq A$, since this union is in V .

Then for every $i < \kappa$, there is $i' \geq i$ such that $x_{i'}$ is a splitting point of T .

Denote by $y_{i'}$ an immediate successor of $x_{i'}$ which is not in b . Let $q_{i'}$ be a condition which witnesses that $y_{i'} \in T$.

Let $C \subseteq \kappa$ be a club such that for every $i_1, i_2 \in C, i_1 < i_2$, we have $i'_1 < i_2$.

The next lemma provides the desired contradiction, since P_κ satisfies κ -c.c.

Lemma 4.3 *The conditions $\{q_{i'} \mid i \in C\}$ are pairwise incompatible.*

Proof. Let $i_1 < i_2$ be in C . Then

$$q_{i'_1} \Vdash \dot{A} \cap \kappa_{i'_1} \neq x_{i'_1+1}.$$

However,

$$q_{i'_2} \Vdash \dot{A} \cap \kappa_{i'_1} = x_{i'_1+1},$$

since $y_{i'_2} >_T x_{i_2} >_T x_{i'_1+1}$. This is possible only when $q_{i'_1}$ and $q_{i'_2}$ are incompatible.

□

Let us give an example of the Easton support iteration P_κ which adds a fresh subset, however we give up here the assumption that $|Q_\beta| < \kappa$.

Let $\langle \kappa_\beta \mid \beta < \kappa \rangle$ be an increasing sequence of measurable cardinals above an inaccessible κ .

Let $\langle P_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton support iterations of the Prikry forcings, i.e. for each $\beta < \kappa$, Q_β is the Prikry forcing with a normal ultrafilter over κ_β .

Let G_κ be a generic subset of P_κ . For every $\beta < \kappa$, let b_β be the Prikry sequence added by G_κ to κ_β .

Lemma 4.4 *The set*

$$A = \{\alpha < \kappa \mid \text{the first element of the sequence } b_{\kappa_\alpha} \text{ is } 0\}$$

is a fresh subset of κ .

Proof. Every initial segment of A is in V due to the support used. On the other hand $A \notin V$, since every condition in the forcing P_κ should be bounded in κ , and so it can be extended to one which forces $b_{\kappa_\alpha}(0) = 0$ or to one forcing $b_{\kappa_\alpha}(0) \neq 0$.

□

Let us turn now to a general case, i.e. we assume only that κ is an inaccessible. Our aim will be to prove the following:

Theorem 4.5 *Let κ be an inaccessible cardinal and $\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be the Easton support iteration of Prikry-type forcing notions. Let I be a stationary subset of κ which consists of singular cardinals μ such that for every $\gamma < \mu$, $|P_\gamma| < \mu$. Suppose that for every $\alpha \in I$, $\Vdash_{P_\alpha} \langle P_\kappa \setminus \alpha, \leq^* \rangle$ is α^{++} -closed.*

Let $G_\kappa \subseteq P_\kappa$ be a generic. Then, in $V[G_\kappa]$, there is no fresh subsets of κ .

Remark 4.6 Similar results were proved in [4] for cardinals above κ . The proof there is based on the fact that $|P_\kappa| = \kappa$ and it is much easier.

Proof. Let G_κ be a generic subset of P_κ .

We would like to show that there is no fresh subset of κ in $V[G_\kappa]$.

Suppose otherwise. Work in V . Let $\mathcal{A}$ be a name of such subset and let $\mathcal{f}$ be a name of the characteristic function of A . Fix some $p \in G_\kappa$ which forces this.

Let $\zeta < \kappa$ be an ordinal for which a new subset is added in the extension from V to $V[G]$. Such ζ exists, since the forcings Q_α (for $\alpha < \kappa$) have cardinality below κ , and at least one of them is non-trivial. By increasing if necessary, we can assume that ζ is a singular cardinal. Then $\langle P_\kappa \setminus \zeta, \leq^* \rangle$ is more than ζ -closed. Thus, there exists a condition $q \in G_\kappa \restriction P_\zeta$ which forces that a new subset is added to ζ . For simplicity, assume that $p \restriction \zeta$ forces this.

Given a condition $r \in P_\kappa$, let us denote by $r(\gamma)$ its γ -th coordinate, i.e. $r = \langle r(\gamma) \mid \gamma < \kappa \rangle$.

We divide into two cases as in Theorem 3.1.

Case 1. There exists $\mu \in (\zeta, \kappa) \cap I$ and a condition $p^* \in P_\mu$ which forces that the following property holds:

$$\begin{aligned} &\exists p \in G_\mu \exists s \in P_\kappa \setminus \mu \forall r \geq^* s \exists \xi < \kappa \exists r_0, r_1 \geq^* r, \\ &V \models (p \restriction r_0 \parallel \mathcal{f} \restriction \xi, p \restriction r_1 \parallel \mathcal{f} \restriction \xi), \text{ and the decisions are different.} \end{aligned}$$

Case 2. For every $\mu \in (\zeta, \kappa) \cap I$, every condition in P_μ forces that–

$$\begin{aligned} &\forall p \in G_\mu \forall s \in P_\kappa \setminus \mu \exists r \geq^* s \forall \xi < \kappa \forall r_0, r_1 \geq^* r, \\ &V \models \text{If } p \restriction r_0 \parallel \mathcal{f} \restriction \xi \text{ and } p \restriction r_1 \parallel \mathcal{f} \restriction \xi \text{ then the decisions are the same.} \end{aligned}$$

The treatment of the first case is exactly as in Theorem 3.1. Let us deal with the second case. As in Theorem 3.1, we define for every $\mu < \kappa$

$$\begin{aligned} e(\mu) = \{ &r \in P_\kappa \setminus \mu : \forall p \in G_\mu \forall \xi < \kappa \forall r_0, r_1 \geq^* r, \\ &V \models \text{If } p \restriction r_0 \parallel \mathcal{f} \restriction \xi \text{ and } p \restriction r_1 \parallel \mathcal{f} \restriction \xi \text{ then the decisions are the same.} \} \end{aligned}$$

By Claim 1 of 3.1, it is $\leq^*$ -dense open subset of $P_\kappa \setminus \mu$, for every $\mu \in (\zeta, \kappa) \cap I$. Again, here is the only place where μ^{++} -closure of the direct order on $P_\kappa \setminus \mu$ is used.

Given a generic $G_\kappa \subseteq P_\kappa, p \in G_\kappa$, define in $V[G_\kappa]$,

$$S = \{\xi < \kappa \mid \xi \text{ is a limit ordinal and } \exists t \geq^* p, t \in G_\kappa \text{ such that } t \restriction \xi \Vdash_{P_\xi} t \restriction \xi \restriction_{P_\kappa \setminus \xi} \dot{A} \cap \xi\}.$$

For every $\xi \in S$, fix some $t^\xi \in G_\kappa$ such that $t^\xi \geq^* p$ and $t^\xi \restriction \xi \Vdash_{P_\xi} t^\xi \restriction \xi \restriction_{P_\kappa \setminus \xi} \dot{A} \cap \xi$. So, there is $p^\xi \in G \restriction P_\xi, p^\xi \geq t^\xi \restriction \xi, (p^\xi) \restriction t^\xi \restriction \xi \restriction_{P_\kappa \setminus \xi} \dot{A} \cap \xi = a_\xi$, for some $a_\xi \in V$. Then there is a finite $b^\xi \subseteq \xi$ such that $p^\xi \restriction b^\xi \geq^* p \restriction (\xi \setminus b^\xi)$.

Suppose for a moment that S is stationary in $V[G_\kappa]$.

Then we can find a stationary subset S' of S and a finite b such that for every $\xi \in S'$, $b^\xi = b$. Now we can freeze $p^\xi \restriction \max(b)$. Denote $\max(b)$ by μ^* .

Let $\mu \in I$ be a cardinal above μ^* .

Consider the set $e(\mu)$ defined above. It is $\leq^*$ -dense open subset of $P_\kappa \setminus \mu$, in $V[G_\mu]$ above $p \restriction \mu$. In particular there is $r \in P_\kappa \setminus \mu, r \geq^* p \restriction \mu$ such that for every $p' \in G_\mu$, for every $\xi < \kappa$ and for every $r_0, r_1 \geq^* r$

$$V \models \text{If } p' \restriction r_0 \restriction_{P_\xi} \dot{A} \cap \xi \text{ and } p' \restriction r_1 \restriction_{P_\xi} \dot{A} \cap \xi \text{ then the decisions are the same.}$$

Recall that $t^\xi \geq^* p$ and $p^\xi \restriction \mu^* \geq^* p \restriction (\mu^*, \xi)$, for every $\xi \in S'$. If we were able to conclude from this that $p^\xi \restriction \mu \geq^* r$, then it will imply that $A \in V$. However it need not be the case since the support of r may be bigger than those of $p \restriction \mu$ and incompatibility may occur on coordinates outside of $\text{supp}(p)$.

Let us argue that it is possible to overcome this obstacle.

Work in V . Set $p_0 = p$. Use Lemma 2.5 to find $q \geq^* p$ and $\mu_0 \geq \sup(\text{supp}(p)), \mu_0 \in I$ such that $q \restriction \mu_0 \restriction_{P_{\mu_0}} \dot{A} \cap \mu_0 \leq \mu_0$.

Then, we use the density of $e(\mu_0)$ to find $r_0 \geq^* q \restriction \mu_0$ such that

$$q \restriction \mu_0 \restriction_{P_{\mu_0}} r_0 \in e(\mu_0).$$

Set $p_1 = q \restriction \mu_0 \restriction_{P_{\mu_0}} r_0$.

Next, we run the argument above with $p = p_0$ replaced by p_1 . Again, using Lemma 2.5 find $q_1 \geq^* p_1$ and $\mu_1 \geq \sup(\text{supp}(p_1)), \mu_0 + 1, \mu_1 \in I$ such that $q_1 \restriction \mu_1 \restriction_{P_{\mu_1}} \dot{A} \cap \mu_1 \leq \mu_1$, where μ^* is now defined using p_1 instead of p .

Then, use the density of $e(\mu_1)$ to find $r_1 \geq^* q_1 \restriction \mu_1$ such that

$$q_1 \restriction \mu_1 \restriction_{P_{\mu_1}} r_1 \in e(\mu_1).$$

Set $p_2 = q \restriction \mu_1 \hat{\smallfrown} r_1$.

Continue by induction and define p_n, q_n, r_n, μ_n , for every $n < \omega$.

Finally set $\mu_\omega = \bigcup_{n < \omega} \mu_n$ and $p_\omega = \bigcup_{n < \omega} p_n = q_\omega = \bigcup_{n < \omega} q_n$. Then, for every $n < \omega$,

$$p_\omega \restriction \mu_n \Vdash p_\omega \restriction \mu_n \in e(\mu_n),$$

since $p_\omega \restriction \mu_n \geq^* r_n \in e(\mu_n)$ and $e(\mu_n)$ is dense open. Also, $\sup(\text{supp}(p_\omega)) = \mu_\omega$.

Pick now a generic $G \subseteq P_\kappa$ with $p_\omega \in G$. Let S_{p_ω} be defined as S above only with p_ω replacing p . Assuming its stationarity, define $p^\xi \geq^* p_\omega, p^\xi \in G \restriction P_\xi$ for $\xi \in S'$ exactly as above. Then there will be a stationary $S' \subseteq S$ and $n^* < \omega$ such that for every $\xi \in S'$, $p^\xi \restriction (\mu_{n^*}, \xi) \geq^* p_\omega \restriction (\mu_{n^*}, \xi)$, since $\sup(\text{supp}(p_\omega)) = \mu_\omega$ and a non-direct extension is used at finitely many places only.

Shrink S' further to S'' and stabilize the value of the function $\xi \mapsto p^\xi \restriction \mu_{n^*}^+$.

Finally, we use that $p_\omega \restriction \mu_{n^*} \in e(\mu_{n^*})$.

Hence, the following lemma will complete the proof. We prove it for the initial p , but the same argument works for S_{p_ω} or any S_x with $x \geq p$.

Lemma 4.7 *S is stationary in $V[G_\kappa]$.*

Proof. The argument will be similar to those of 2.2.

Suppose otherwise. Let $C \subseteq \kappa$ be a club disjoint from S . Assume that $p \in G_\kappa$ forces this, otherwise replace it by a stronger condition doing this.

Work in V . Pick an elementary submodel $M \preceq H_\chi$ such that

1. $|M| = \delta < \kappa$,
2. $M \cap \kappa = \delta$,
3. $\text{cof}(\delta) < \delta$,
4. ${}^{\text{cof}(\delta)}M \subseteq M$,
5. $\kappa, P_\kappa, \mathcal{C}, p, I \in M$.

Pick a cofinal in δ sequence $\langle \delta_i \mid i < \text{cof}(\delta) \rangle$ consisting of singulars and with $\delta_0 > \text{cof}(\delta)$. Consider—

$$D^0 = \{r \geq p \mid \exists c < \kappa \quad r \Vdash c = \min(\mathcal{C} \setminus \delta_0)\}.$$

Clearly, D^0 is a dense open and it belongs to M .

Apply Lemma 2.3. Then there will be $\alpha_0 < \kappa$ and $q_0 \geq^* p, q_0 \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$,

in M such that for every $r \in P_{\alpha_0}, r \geq_{P_{\alpha_0}} q_0 \restriction \alpha_0$ there is $r' \geq_{P_{\alpha_0}} r$ such that $r' \restriction q_0 \setminus \alpha_0 \in D_0$. So, for every such r' there is $c(r') < \kappa$ such that

$$r' \restriction q_0 \setminus \alpha_0 \Vdash c(r') = \min(\mathcal{C} \setminus \delta_0).$$

Note that $P_{\alpha_0} \subseteq M$. Hence all $c(r')$'s are in M . Also $|P_{\alpha_0}| < \kappa$. Hence their sup is below κ , and then, by elementarity, in M . Denote it by c_0^* . So,

$$q_0 \Vdash \min(\mathcal{C} \setminus \delta_0) < c_0^*.$$

Define, for every $\tau < \kappa$,

$$D_\tau = \{r \in P \mid r \geq^* p \text{ and } \exists \xi \in \kappa \setminus \tau \quad r \restriction \xi \Vdash_{P_\xi} (r \restriction \xi) \parallel_{P_{>\xi}} \mathcal{A} \cap \xi\}.$$

Claim 3 D_τ is $\leq^*$ -dense open above p .

Proof. Set $\tau_0 = \tau + 1$. Consider

$$D(\tau_0) = \{r \in P_\kappa \mid r \perp p \text{ or } (r \geq p \text{ and } r \parallel \mathcal{A} \cap \tau_0)\}.$$

By Lemma 2.3, there are α_0 and $q \geq^* p$ such that

$$q \restriction \alpha_0 \Vdash \exists b \in \mathcal{G}_{\alpha_0} \quad b \restriction q \setminus \alpha_0 \parallel \mathcal{A} \cap \tau_0.$$

If $\alpha_0 \leq \tau_0$ then we are done. Suppose that $\alpha_0 > \tau_0$. Consider $D(\alpha_0)$ and again, using Lemma 2.3, pick α_1 and $q_1 \geq^* q$ such that

$$q_1 \restriction \alpha_1 \Vdash \exists b \in \mathcal{G}_{\alpha_1} \quad b \restriction q_1 \setminus \alpha_1 \parallel \mathcal{A} \cap \alpha_0.$$

If $\alpha_1 \leq \alpha_0$, then $q_1 \in D(\alpha_0)$ and we are done. If $\alpha_1 > \alpha_0$, then continue and define in the same fashion α_2, q_2 etc.

Suppose that the process continues infinitely many steps. Then we will have

$$\alpha_0 < \alpha_1 < \dots < \alpha_i < \dots \text{ and } q_0 \leq^* q_1 \leq^* \dots \leq^* q_i \leq^* \dots, i < \omega.$$

Let $\alpha^* = \bigcup_{i < \omega} \alpha_i$ and $q^* \geq^* q_i$, for every $i < \omega$. Let $G \subseteq P_\kappa$ be a generic with $q^* \in G$. For every $i < \omega$, let $G_{\alpha_i} = G \restriction P_{\alpha_i}$ and $G_{\alpha^*} = G \restriction P_{\alpha^*}$.

Now, for every $i < \omega$, there are $r_{i+1} \in G_{\alpha_{i+1}}$ and $a_i \in V$ such that

$$r_{i+1} \restriction q_{i+1} \setminus \alpha_{i+1} \Vdash \mathcal{A} \cap \alpha_i = a_i.$$

Therefore,

$$r_{i+1} \frown q^* \setminus \alpha_{i+1} \Vdash \mathcal{A} \cap \alpha_i = a_i.$$

Set $a = \bigcup_{i < \omega} a_i$. Then, in $V[G_{\alpha^*}]$, $q^* \setminus \alpha^* \Vdash \mathcal{A} \cap \alpha^* = a$, since each condition of the form $r_{i+1} \frown q \upharpoonright [\alpha_{i+1}, \alpha^*)$ belongs to G_{α^*} . So, there is $r \in G_{\alpha^*}$ such that

$$r \frown q^* \setminus \alpha^* \Vdash \mathcal{A} \cap \alpha^* = a.$$

Hence,

$$r \Vdash_{P_{\alpha^*}} (q^* \setminus \alpha^* \parallel \mathcal{A} \cap \alpha^*).$$

The only requirement on G_{α^*} was that $q^* \upharpoonright \alpha^* \in G_{\alpha^*}$. Hence,

$$q^* \upharpoonright \alpha^* \Vdash q^* \setminus \alpha^* \parallel \mathcal{A} \cap \alpha^*$$

and so $q^* \in D_\tau$.

□ of the claim.

Consider $D_{c_0^*}$. It is in M , as well. So, inside M , we can pick $\xi_0 \geq c_0^*$, $\xi_0 \in I$ and $t_1 \geq^* q_0$ such that $t_0 \upharpoonright \text{cf}(\delta) + 1 = p \upharpoonright \text{cf}(\delta) + 1$, and—

$$t_0 \upharpoonright \xi_0 \Vdash_{P_{\xi_0}} t_0 \setminus \xi_0 \parallel_{P_\kappa \setminus \xi_0} \mathcal{A} \cap \xi_0$$

and—

$$t_0 \Vdash \min(C \setminus \delta_0) < \xi_0.$$

We continue the same process and construct a $\leq^*$ increasing sequence $\langle t_i : i \leq \text{cf}(\delta) \rangle$ and $\langle \xi_i : i \leq \text{cf}(\delta) \rangle$ such that each strict initial segments of the sequences is in M , and—

- $t_i \upharpoonright \text{cf}(\delta) + 1 = p \upharpoonright \text{cf}(\delta) + 1$.
- $t_i \upharpoonright \xi_i \Vdash_{P_{\xi_i}} t_i \setminus \xi_i \parallel_{P_\kappa \setminus \xi_i} \mathcal{A} \cap \xi_i$.
- $t_i \Vdash \min(C \setminus \delta_i) \leq \xi_i$ (furthermore, for each successor i , $t_i \Vdash \min(C \setminus \delta_i) < \xi_i$).

The successor stages in the construction are as above: first, $q_{i+1} \geq^* t_i$ and $c_{i+1}^* > \xi_i$ are constructed in M , such that $q_{i+1} \upharpoonright \text{cf}(\delta) + 1 = p \upharpoonright \text{cf}(\delta) + 1$ and $q_{i+1} \Vdash \min(C \setminus \delta_{i+1}) < c_{i+1}^*$. Then, $t_{i+1} \geq^* q_{i+1}$ is constructed inside M , such that $t_{i+1} \upharpoonright \text{cf}(\delta) + 1 = p \upharpoonright \text{cf}(\delta) + 1$, and, for some $\xi_{i+1} \in I \setminus c_{i+1}^*$,

$$t_{i+1} \upharpoonright \xi_{i+1} \Vdash_{P_{\xi_{i+1}}} t_{i+1} \setminus \xi_{i+1} \parallel_{P_\kappa \setminus \xi_{i+1}} \mathcal{A} \cap \xi_{i+1}.$$

At limit stages of the construction and at the stage $\text{cof}(\delta)$ itself we would like to take an upper bound of the sequences constructed so far. So, assuming that $i^* \leq \text{cof}(\delta)$ is limit and $\langle t_i : i < i^* \rangle$, $\langle \xi_i : i < i^* \rangle$ were constructed, let t_{i^*} be a $\leq^*$ -upper bound of $\langle t_i : i < i^* \rangle$ and $\xi_{i^*} = \sup\{\xi_i : i < i^*\}$. Taking an upper bound of the sequence of conditions is possible since for each $i < \text{cof}(\delta)$, $p_i \restriction \text{cof}(\delta) + 1 = p \restriction \text{cof}(\delta) + 1$, and $\langle P_\kappa \setminus \text{cof}(\delta) + 1, \leq^* \rangle$ is more than $\text{cof}(\delta)$ -closed. The fact that ${}^{<\text{cof}(\delta)}M \subseteq M$ implies that each strict initial segment of $\langle t_i \mid i < \text{cof}(\delta) \rangle$, $\langle \xi_i : i < \text{cof}(\delta) \rangle$ belongs to M (but the entire sequences, and the limit $t^* = t_{\text{cof}(\delta)}$, are external to M).

Finally, letting $t^* = t_{\text{cof}(\delta)}$, we have $t^* \Vdash \delta \in \mathcal{C}$, since $\mathcal{C}$ is forced to be closed by p .

Let $G \subseteq P_\kappa$ be a generic with $t^* \in G$. Set $G_\delta = G \restriction P_\delta$. Then for every $i < \text{cof}(\delta)$, $t_i \restriction \delta \in G_\delta$. Hence, there are $r_i \in G_{\xi_i}$, $r_i \geq t_i \restriction \xi_i$ and $a_i \in V$ such that

$$r_i \restriction t_i \setminus \xi_i \Vdash \dot{A} \cap \xi_i = a_i.$$

Set $a = \bigcup_{i < \text{cof}(\delta)} a_i$. Then, in $V[G]$, $A \cap \delta = a$, since each $r_i \restriction t_i \setminus \xi_i$ is in G .

Remember that the only requirement on a generic set G was that t^* belongs to it. Hence, back in V ,

$$t^* \restriction \delta \Vdash_{P_\delta} t^* \setminus \delta \Vdash_{P_{>\delta}} \dot{A} \cap \delta.$$

So, t^* forces $\delta \in \mathcal{G}$, and, as was shown above, $\delta \in \mathcal{C}$ as well. Contradiction.

□

5 No fresh subsets of κ in the full support

Our aim will be to prove the following :

Theorem 5.1 *Let κ be an inaccessible cardinal and $\langle P_\alpha, \mathcal{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be the full support iteration of Prikry-type forcing notions. Suppose that for every $\beta < \kappa$, for every $x, y, z \in \mathcal{Q}_\beta$, if $z \leq^*_{\mathcal{Q}_\beta} x, y$ and x, y are compatible according to $\leq_{\mathcal{Q}_\beta}$, then they are compatible according to $\leq^*_{\mathcal{Q}_\beta}$, i.e. there is $e \in \mathcal{Q}_\beta$, $e \geq^*_{\mathcal{Q}_\beta} x, y$.¹¹*

Let $G_\kappa \subseteq P_\kappa$ be a generic. Then, in $V[G_\kappa]$, there is no fresh subsets of κ .

Remark 5.2 More restrictive results were proved in [4] for cardinals above κ . The present proof can be easily modified for higher cardinals.

¹¹Note that if $\leq = \leq^*$, then this holds trivially. Prikry, Magidor, Radin forcings, their supercompact versions, etc., have this property. Actually, any reasonable forcing of this type has this property.

Proof. Let G_κ be a generic subset of P_κ .

We would like to show that there is no fresh subset of κ in $V[G_\kappa]$.

Suppose otherwise. Work in V . Let $\tilde{A}$ be a name of such subset and let $\tilde{f}$ be a name of the characteristic function of A . Fix some $p \in G_\kappa$ which forces this.

Let $\zeta < \kappa$ be an ordinal for which a new subset is added in the extension from V to $V[G]$. Such ζ exists, since the forcings Q_α (for $\alpha < \kappa$) have cardinality below κ , and at least one of them is non-trivial. By increasing if necessary, we can assume that ζ is a singular cardinal. Then $\langle P_\kappa \setminus \zeta, \leq^* \rangle$ is more than ζ -closed. Thus, there exists a condition $q \in G_\kappa \restriction P_\zeta$ which forces that a new subset is added to ζ . For simplicity, assume that $p \restriction \zeta$ forces this.

Let I be a subset of κ which consists of singular cardinals τ such that for every $\rho < \tau$, $|P_\rho| < \tau$.

As before, we divide into two cases:

Case 1. There exists $\mu \in (\zeta, \kappa) \cap I$ and a condition $p^* \in P_\mu$, $p^* \geq p \restriction \mu$ which forces the following property:

There are $p' \in G_\mu$ and $s \in P_\kappa \setminus \mu$ such that $s \geq p \setminus \mu$, and, for every $r \geq^* s$, if $r(\mu) = s(\mu)$, $r(\mu^+) = s(\mu^+)$, then the following holds: there are $\xi < \kappa$ and $r_0, r_1 \geq^* r$ such that $r_0(\mu) = r_1(\mu) = r(\mu)$, $r_0(\mu^+) = r_1(\mu^+) = r(\mu^+)$, and, additionally, $V \models (p' \restriction r_0 \parallel \tilde{f} \restriction \xi, p' \restriction r_1 \parallel \tilde{f} \restriction \xi)$, and the decisions are different.

Above, G_μ denotes the canonical name for the generic set for P_μ .

By extending p^* , if necessary, we can decide the value of p' in the statement above, and thus assume that $p^* \geq p'$. Let $\tilde{s}$ be a P_μ -name for s from the above property, and assume that this is forced by p^* .

Note that here we do not assume μ^{++} -completeness of the direct order. Additional requirements are included in order to compensate this. Still, the treatment of this case repeats completely Case 1 of 3.1.

Case 2. For every $\mu \in (\zeta, \kappa) \cap I$, every condition in P_μ stronger than $p \restriction \mu$ forces (and so, $p \restriction \mu$ forces) that–

For every $p' \in G_\mu$ and $s \in P_\kappa \setminus \mu$ such that $s \geq p \setminus \mu$, there exists $r \geq^* s$ with $r(\mu) = s(\mu)$, $r(\mu^+) = s(\mu^+)$, such that for every $\xi < \kappa$ and $r_0, r_1 \geq^* r$, if $r_0(\mu) = r_1(\mu) = r(\mu)$, $r_0(\mu^+) = r_1(\mu^+) = r(\mu^+)$, then $V \models$ If $p' \restriction r_0 \parallel \tilde{A} \restriction \xi$ and $p' \restriction r_1 \parallel \tilde{A} \restriction \xi$ then the decisions are the same.

For every $\mu \in (\zeta, \kappa) \cap I$, define (in $V[G_\mu]$)

$$\begin{aligned} e'(\mu) = \{ & r \in P_\kappa \setminus \mu : \text{for every } p' \in G_\mu, \xi < \kappa \text{ and } r_0, r_1 \geq^* r, \\ & \text{if } r_0(\mu) = r_1(\mu) = r(\mu), r_0(\mu^+) = r_1(\mu^+) = r(\mu^+), \text{ then} \\ & V \models \text{If } p' \frown r_0 \parallel \dot{f} \restriction \xi \text{ and } p' \frown r_1 \parallel \dot{f} \restriction \xi \text{ then the decisions are the same.} \} \end{aligned}$$

Claim 4 For every $\mu \in (\zeta, \kappa) \cap I$,

$$e'(\mu) \text{ is } \leq^* \text{-dense in } P \setminus \mu \text{ (above } p \setminus \mu).^{12}$$

Proof. Note that $|G_\mu| \leq \mu^+$, the forcing $\langle P_\kappa \setminus \mu^{++} \rangle$ is μ^{++} -closed and the coordinates μ, μ^+ do not change. Hence, for every given $r' \in P_\kappa \setminus \mu$, we can construct a $\leq^*$ -increasing sequence of a length $|G_\mu|$ of conditions stronger than r' , which takes care of each $p' \in G_\mu$. Then its upper bound will be in $e(\mu)$.

□ of the claim.

Given a generic $G_\kappa \subseteq P_\kappa, p \in G_\kappa$, define in $V[G_\kappa]$,

$$S = \{ \xi < \kappa \mid \xi \text{ is a limit ordinal and } \exists t \geq^* p, t \in G_\kappa \text{ such that } t \restriction \xi \Vdash_{P_\xi} t \setminus \xi \parallel_{P_\kappa \setminus \xi} \dot{A} \cap \xi \}.$$

For every $\xi \in S$, fix some $t^\xi \in G_\kappa$ such that $t^\xi \geq^* p$ and $t^\xi \restriction \xi \Vdash_{P_\xi} t^\xi \setminus \xi \parallel_{P_\kappa \setminus \xi} \dot{A} \cap \xi$. So, there is $p^\xi \in G \restriction P_\xi, p^\xi \geq t^\xi \restriction \xi, (p^\xi)^\frown t^\xi \setminus \xi \Vdash \dot{A} \cap \xi = a_\xi$, for some $a_\xi \in V$. Then there is a finite $b^\xi \subseteq \xi$ such that $p^\xi \setminus b^\xi \geq^* p \restriction (\xi \setminus b^\xi)$.

The argument of Lemma 4.7 applies without changes in the present situation, and shows that S is stationary (in $V[G_\kappa]$).

Then we can find a stationary subset S' of S and a finite b such that for every $\xi \in S'$, $b^\xi = b$. Let $\mu \in I$ be any cardinal above $\max(b)$. By shrinking S' , we can freeze $p^\xi \restriction \mu$.

So there exists a condition $q \in G_\kappa \cap P_\mu$ extending $p \restriction \mu$, and, for every $\xi \in S'$, q forces that there exists a direct extension $r^\xi \geq^* p \setminus \mu$ such that $q \frown r^\xi$ decides $\dot{A} \cap \xi$ (actually, we can take $r^\xi = p^\xi \setminus \mu \frown t^\xi \setminus \xi$ and also get that $q \frown r^\xi \in G_\kappa$).

If the following holds,

($\aleph$): For every $\beta < \kappa$, for every $s, t, r \in \mathcal{Q}_\beta$, if $s \leq_{\mathcal{Q}_\beta}^* t, r$, then there is $e \in \mathcal{Q}_\beta, e \geq_{\mathcal{Q}_\beta}^* t, r$.¹³

then we are done: indeed, $q \frown p \setminus \mu$ forces that—

$$\dot{A} = \bigcup_{\xi < \kappa} \{ X \subseteq \xi : X \in V, \exists \dot{r}, q \Vdash \dot{r} \geq^* p \setminus \mu, q \frown \dot{r} \Vdash X = \dot{A} \cap \xi \},$$

¹²It need not be open.

¹³For example if P_κ is the Magidor iteration of Prikry forcings. Also note that we do not need to split into Cases 1,2 in this type of situation.

which is a contradiction since the set to the right belongs to V .

Let us deal with the general case. We assumed the following,

($\beth$): For every $\beta < \kappa$, for every $x, y, z \in \mathcal{Q}_\beta$, if $z \leq_{\mathcal{Q}_\beta}^* x, y$ and x, y are compatible according to $\leq_{\mathcal{Q}_\beta}$, then they are compatible according to $\leq_{\mathcal{Q}_\beta}^*$, i.e. there is $e \in \mathcal{Q}_\beta$, $e \geq_{\mathcal{Q}_\beta}^* x, y$.

Now, P_κ is a full support iteration. So, ($\beth$) implies -

($\beth$): For every $u, v, w \in P_\kappa$, if $w \leq_{P_\kappa}^* u, v$ and u, v are compatible according to $\leq_{P_\kappa}$, then they are compatible according to $\leq_{P_\kappa}^*$.¹⁴

By shrinking S' , we can assume that for every $\xi, \xi' \in S'$, $r^\xi \restriction \mu^{++} = r^{\xi'} \restriction \mu^{++}$. Denote by w the stabilized value. Assume that $s \geq w \restriction p \setminus (\mu^+ + 1)$ is a condition forcing the above picture. In other words, s forces that for some stationary $\mathcal{S} \subseteq \kappa$, and for every $\xi \in \mathcal{S}$, there exists a condition $t^\xi \in \mathcal{G}$, $t^\xi \geq^* p$, and a condition $p^\xi \geq t^\xi \restriction \xi$ such that $p^\xi \restriction t^\xi \setminus \xi \restriction \mathcal{A} \cap \xi$, and $p^\xi \restriction t^\xi \setminus \xi \restriction \mu^+ + 1 = w$.

Let $r \in P \setminus \mu$, $r \geq^* s \restriction \mu$ be a condition such that $s \restriction \mu \Vdash r \in e'(\mu)$. We argue that $s^* = s \restriction \mu \restriction r$ forces that $\mathcal{f} \in V$. Let-

$$g = \bigcup_{\xi < \kappa} \{g \in 2^\xi : X \in V, \exists u \geq^* s^* \restriction \mu^+ + 1, (s^* \restriction \mu^+ + 1) \restriction u \Vdash g = \mathcal{f} \restriction \xi\}.$$

Note that since $s^* \restriction \mu \in e'(\mu)$, g is a function. Furthermore,

$$s^* \Vdash \mathcal{f} = g.$$

Indeed, assume that $G' \subseteq P_\kappa$ is a generic containing s^* . Let $\xi \in S'$. Then there exists some $u \geq^* p \restriction (\mu^+ + 1)$ in $G' \cap P_\kappa \restriction \mu^+ + 1$ such that $(s \restriction \mu^+ + 1) \restriction u \restriction \mathcal{f} \restriction \xi$ and $(s \restriction \mu^+ + 1) \restriction u \in G'$ (since this was forced by s). By ($\beth$), $u, r \restriction (\mu^+ + 1)$ are $\leq^*$ -compatible. Let $u^* \geq^* u, r \restriction (\mu^+ + 1)$. Then $(s^* \restriction \mu^+ + 1) \restriction u^*$ witnesses the fact that $\mathcal{f} \restriction \xi = g \restriction \xi$. Since this is true for every $\xi \in S'$, we get $\mathcal{f} = g \in V$. $\square$

Lemma 5.3 *Under the assumptions of the previous Theorem, $P = P_\kappa$ does not add fresh unbounded subsets to any λ with $\text{cof}(\lambda) > \kappa$.*

Proof. Let $A \in V[G_\kappa]$ be a fresh unbounded subset of λ with a characteristic function f . Proceed as in the proof of Theorem 5.1. Case 1 is now formulated as follows¹⁵:

¹⁴It need not be the case for Easton or non-stationary support, since then the support of w may be strictly smaller than those of u, v , and u, v may disagree on a common coordinate outside the support of w .

¹⁵The only difference is that " $\exists \xi < \kappa$ " is replaced with " $\exists \xi < \lambda$ ".

Case 1. There exists $\mu \in (\zeta, \kappa) \cap I$ and a condition $p^* \in P_\mu, p^* \geq p \restriction \mu$ which forces the following property:

There are $p' \in \mathcal{G}_\mu$ and $s \in P_\kappa \setminus \mu$ such that $s \geq p \restriction \mu$, and, for every $r \geq^* s$, if $r(\mu) = s(\mu), r(\mu^+) = s(\mu^+)$, then the following holds: there are $\xi < \lambda$ and $r_0, r_1 \geq^* r$ such that $r_0(\mu) = r_1(\mu) = r(\mu), r_0(\mu^+) = r_1(\mu^+) = r(\mu^+)$, and, additionally, $V \models (p' \restriction r_0 \parallel \mathcal{f} \restriction \xi, p' \restriction r_1 \parallel \mathcal{f} \restriction \xi)$, and the decisions are different.

The treatment of case 1 remain the same. Let us concentrate on Case 2, which is the following:

Case 2. For every $\mu \in (\zeta, \kappa) \cap I$, every condition in P_μ stronger than $p \restriction \mu$ forces (and so, $p \restriction \mu$ forces) that–

For every $p' \in \mathcal{G}_\mu$ and $s \in P_\kappa \setminus \mu$ such that $s \geq p \restriction \mu$, there exists $r \geq^* s$ with $r(\mu) = s(\mu), r(\mu^+) = s(\mu^+)$, such that for every $\xi < \lambda$ and $r_0, r_1 \geq^* r$, if $r_0(\mu) = r_1(\mu) = r(\mu), r_0(\mu^+) = r_1(\mu^+) = r(\mu^+)$, then $V \models$ If $p' \restriction r_0 \parallel \mathcal{A} \cap \xi$ and $p' \restriction r_1 \parallel \mathcal{A} \cap \xi$ then the decisions are the same.

As before, for every $\mu \in (\zeta, \kappa) \cap I$, $p \restriction \mu$ forces that the following set is $\leq^*$ -dense in $P_\kappa \setminus \mu$:

$$e'(\mu) = \{r \in P_\kappa \setminus \mu: \text{ for every } p' \in \mathcal{G}_\mu, \xi < \lambda \text{ and } r_0, r_1 \geq^* r, \\ \text{ if } r_0(\mu) = r_1(\mu) = r(\mu), r_0(\mu^+) = r_1(\mu^+) = r(\mu^+), \text{ then} \\ V \models \text{ If } p' \restriction r_0 \parallel \mathcal{f} \restriction \xi \text{ and } p' \restriction r_1 \parallel \mathcal{f} \restriction \xi \text{ then the decisions are the same.}\}$$

Now proceed in the following way. For every $\xi < \lambda$, let $p^\xi \in G_\kappa$ be an extension of p which decides $\mathcal{A} \cap \xi$. Let $b^\xi \subseteq \kappa$ be a finite set such that for every $\alpha \notin b^\xi, p^\xi \restriction \alpha \Vdash p^\xi(\alpha) \geq^*_{\mathcal{Q}_\alpha} p(\alpha)$. Since $\text{cof}(\lambda) > \kappa$, we may find an unbounded $S' \subseteq \kappa^+$ and a finite $b \subseteq \kappa$, such that for every $\xi \in S', b^\xi = b$. Let $\mu \in (\zeta, \kappa) \cap I$ be a cardinal above $\max(b)$.

Thus, there is $S' \subseteq \lambda$ stationary, such that for every $\xi \in S'$, a condition of the form $p^\xi \restriction \mu \restriction p^\xi \restriction \mu \in G_\kappa$ decides $\mathcal{f} \restriction \xi$ and has the property that $p^\xi \restriction \mu \geq^* p \restriction \mu$. The stationary set S' replaces the set S' from the second case of Theorem 5.1, and, from here, the proof is completed exactly as there. $\square$

6 The Approximation and cover Properties

Definition 6.1 (Hamkins [8]) Let δ be a regular uncountable cardinal, and let $N \subseteq V$ be a transitive inner model containing the ordinals.

1. N has the δ -cover property if for every $A \in V$, $A \subseteq N$ such that $|A| < \delta$, there exists $B \in N$ with $|B|^N < \delta$ such that $A \subseteq B$.
2. N has the δ -approximation property if for every $A \in V$, $A \subseteq N$, the following are equivalent:
 - (a) $A \in N$.
 - (b) A is δ -approximated in N : that is, for every $X \in N$ with $|X|^N < \delta$, $A \cap X \in N$.

The properties were introduced by Hamkins. One of the central application of the properties is the absorption of large cardinals of V above κ into N , whenever $N \subseteq V$ is an inner model of V with the κ -cover and approximation properties (see [8] for a detailed discussion about that). One example of this phenomenon is the following Lemma.

Lemma 6.2 *Let κ be a regular uncountable cardinal. Assume that $N \subseteq V$ as above has the κ approximation property. Then every κ -complete ultrafilter $W \in V$ whose underlying set is some $X \in N$, extends a κ -complete ultrafilter of N . In other words, for every such W , $W \cap N \in N$.*

Proof. In N , fix an enumeration $\langle X_\alpha : \alpha < (2^{|X|})^N \rangle$ of the powerset of X . Let $A = \{\alpha < (2^{|X|})^N : X_\alpha \in W\}$. It suffices to prove that $A \in N$, and, by the κ -approximation property, it suffices to prove that A is κ -approximated over N . Indeed, assume that $B \subseteq (2^{|X|})^N$ has size less than κ , and let us argue that $A \cap B \in N$. Since $W \in V$ is κ -complete, the set–

$$\left(\bigcap_{\alpha \in B, X_\alpha \in W} X_\alpha \right) \cap \left(\bigcap_{\alpha \in B, X_\alpha \notin W} X \setminus X_\alpha \right)$$

belongs to W , and in particular non-empty. Pick x in this set. Then $A \cap B = \{\alpha \in B : x \in X_\alpha\}$, and this definition is carried out in N . $\square$

Lemma 6.2 is often referred to as "weak universality". We remark that the κ -cover and approximation properties imply a stronger form of universality that involves extenders, which is referred to as "Woodin's Universality Theorem":

Theorem 6.3 (Woodin) *Let κ be a regular uncountable cardinal. Assume that the extension $N \subseteq V$ as above satisfies the κ -cover and approximation properties and $E \in V$ is a (κ, λ) -extender. Then $E \cap N \in N$,¹⁶ provided that, for every $A \in \mathcal{P}(\lambda) \cap N$, $j_E(A) \cap \lambda \in N$.*

¹⁶We view E as the sequence $\langle (a, A) : a \in [\lambda]^{<\omega} \text{ and } a \in j_E(A) \rangle$, so $E \cap N$ is the N -extender whose derived measures are the restrictions of the derived measures of E .

Goldberg asked whether V has the κ -cover and approximation properties inside $V[G]$, where $G \subseteq P_\kappa$ is generic over V (this is Question 1.7; P_κ is as in Subsection 1.2). Our goal in this section will be to provide an affirmative answer, see Theorem 6.6.

Lemma 6.4 (Goldberg) *Let κ be regular. Assume that $V \subseteq V'$ is a cardinal preserving extension which has the κ^+ -cover property, and does not add fresh subsets to κ . Then it does not add fresh subsets to any ordinal of cofinality (in V) κ .*

Proof. Assume that $\text{cf}(\alpha) = \kappa$ and $A \subseteq \alpha$ is fresh over V . Fix in V an increasing continuous cofinal sequence $\langle \alpha_\xi : \xi < \kappa \rangle$ in α . The set $Y = \{A \cap \alpha_\xi : \xi < \kappa\} \in V'$ has size κ . By freshness of A , $Y \subseteq V$. By the κ^+ -approximation property, there exists a set $X \in V$ of size κ that covers Y . In other words, for every $\xi < \kappa$, $A \cap \alpha_\xi \in X$. By shrinking X , if necessary, we can assume that every element of X is a bounded subset of α .

Enumerate $X = \langle x_\nu : \nu < \kappa \rangle \in V$. Then the set $B = \{\nu < \kappa : \exists \xi < \kappa, A \cap \alpha_\xi = x_\nu\}$ is a fresh subset of κ . Indeed, for every $\nu^* < \kappa$, let $\xi^* < \kappa$ be a limit ordinal, such that $\sup\{\sup(x_\nu) : \nu < \nu^*\} < \alpha_{\xi^*}$. Since $A \cap \alpha_{\xi^*} \in V$, it can be used to define in V the set $B \cap \nu^* = \{\nu < \nu^* : \exists \xi < \xi^*, x_\nu = (A \cap \alpha_{\xi^*}) \cap \alpha_\xi\}$. So each strict initial segment of B belongs to V . However, $B \notin V$ since $A = \bigcup_{\nu \in B} x_\nu$ would belong to V if B belonged to V . So B is a fresh subset of κ over V , which is a contradiction. $\square$

We proceed and prove the κ -cover property. We will actually prove a bit more - the κ^+ -cover property.

Theorem 6.5 *Let $P = P_\kappa$ be as in subsection 1.2. Let $G \subseteq P$ be generic over V . Then:*

1. *V satisfies the κ -cover property in $V[G]$.*
2. *If the Easton or nonstationary support are taken, V satisfies the κ^+ -cover property in $V[G]$.*
3. *If the Full support is taken, V satisfies the κ^+ -cover property in $V[G]$, provided that for every $\alpha < \kappa$, $\langle \mathcal{Q}_\alpha, \leq_{\mathcal{Q}_\alpha}, \leq_{\mathcal{Q}_\alpha}^* \rangle$ satisfies the following property: for every $x, y, z \in \mathcal{Q}_\alpha$, if $z \leq_{\mathcal{Q}_\alpha}^* x, y$ then x, y are $\leq^*$ -compatible.*

Proof.

1. We prove that V has the κ -cover property. Let $A \in V[G]$ be a set of ordinals of size $< \kappa$. Assume that $\zeta < \kappa$ and $\langle a_\xi : \xi < \zeta \rangle$ is a sequence of P -names for an enumeration

of $\mathcal{A}$ whose order type ζ is forced, by the weakest condition, to be below κ . By Lemma 2.5, there exists $p \in G$ and $\zeta^* < \kappa$ such that $p \Vdash \zeta < \zeta^*$. Assume for simplicity that p is the weakest condition (else, work above it). Work in $V[G \cap P_{\zeta^*+1}]$ and construct a $\leq^*$ -increasing sequence $\langle q_\xi : \xi < \zeta^* \rangle$ of conditions in $P \setminus \zeta^* + 1$, and a sequence of sets of ordinals of size $< \kappa$, $\langle A_\xi : \xi < \zeta^* \rangle$ such that for each $\xi < \zeta^*$, $q_\xi \Vdash \mathcal{A}_\xi \in A_\xi$ (where $\mathcal{A}_\xi$ is defined to be 0 in the case where $\xi > \zeta$). Note that the construction is done by a successive use of Lemma 2.5, and we have enough closure at limit steps. Finally, let q^* be the coordinatewise supremum of $\langle q_\xi : \xi < \zeta^* \rangle$, and $A^* = \bigcup \{A_\xi : \xi < \zeta^*\}$. Then A^* has size $< \kappa$. So $q^* \in P \setminus \zeta^* + 1$ and $q^* \Vdash \mathcal{A} \subseteq A^*$, but q^*, A^* were constructed in $V[G \cap P_{\zeta^*+1}]$; since P_{ζ^*+1} has size $< \kappa$, we can find $p \in P_\kappa$ and A^{**} in V such that $|A^{**}| < \kappa$ and $p \Vdash \mathcal{A} \subseteq A^{**}$.

2. If the Easton support is taken, then $|P| = \kappa$ and generic extensions with P have the κ^+ -cover property (as any other extension with a κ^+ -c.c. forcing). Thus, let us take care of the nonstationary support case. Let $\mathcal{A}$ be a P -name for a set of size κ , and pick P -names $\langle \mathcal{A}_\xi : \xi < \kappa \rangle$ for an enumeration of $\mathcal{A}$. Following the standard fusion argument, we construct sequences:

- $\langle p_\xi : \xi < \kappa \rangle$ an $\leq^*$ -increasing sequence of conditions.
- $\langle \nu_\xi : \xi < \kappa \rangle$ a continuous increasing sequence of ordinals in κ .
- $\langle C_\xi : \xi < \kappa \rangle$ a decreasing sequence of clubs in κ , each C_ξ disjoint from the support of p_ξ .
- $\langle A_\xi : \xi < \kappa \rangle$ a sequence of sets of size $< \kappa$ in V , such that, for each $\xi < \kappa$, $p_\xi \Vdash \mathcal{A}_\xi \in A_\xi$.

We make sure during the construction that for every $\xi' < \xi$, $p_\xi \restriction \nu_{\xi'} + 1 = p_{\xi'} \restriction \nu_{\xi'} + 1$, and $\nu_\xi \in C_{\xi'}$.

The construction is a standard fusion construction. Assuming $p_\xi, \nu_\xi, C_\xi, A_\xi$ were chosen, let $\nu_{\xi+1}$ be the least point of C_ξ above ν_ξ . Let $p_{\xi+1}$ be a condition such that $p_{\xi+1} \restriction \nu_{\xi+1} + 1 = p_\xi \restriction \nu_{\xi+1} + 1$, $\nu_{\xi+1} \notin \text{supp}(p_{\xi+1})$, and $p_{\xi+1} \setminus \nu_{\xi+1} + 1$ decides the value of $\mathcal{A}_\xi$ up to $< \kappa$ possibilities (using Lemma 2.5), where the set of those possibilities belongs to $V[G \cap P_{\xi+1}]$; as usual, we can assume that for some $A_\xi \in V$ of size $< \kappa$, $p_{\xi+1} \Vdash \mathcal{A}_\xi \in A_\xi$, since $P_{\xi+1}$ has size below κ . Finally, let $C_{\xi+1}$ be a sub-club of C_ξ disjoint from the support of $p_{\xi+1}$.

In limit steps, take $\nu_\xi = \bigcup_{\xi' < \xi} \nu_{\xi'}$ and p_ξ that satisfies $p_\xi \restriction \nu_\xi = \bigcup_{\xi' < \xi} p_{\xi'} \restriction \nu_{\xi'} + 1$,

$\nu_\xi \notin \text{supp}(p_\xi)$, $p_\xi \setminus \nu_\xi + 1$ is $\leq^*$ above the pointwise supremum of $\langle p_{\xi'} \setminus \nu_\xi + 1 : \xi' < \xi \rangle$ and $p_\xi \Vdash \dot{a}_\xi \in A_\xi$ for some A_ξ in V of size $< \kappa$.

This concludes the construction. Finally, let $p^* = \bigcup_{\xi < \kappa} p_\xi \restriction \nu_\xi + 1$ and $A^* = \bigcup_{\xi < \kappa} A_\xi$. Then $p^* \Vdash \dot{A} \subseteq A^*$ and $|A^*| \leq \kappa$.

3. It suffices to prove that the additional assumption imposed on P ensures that P has the κ^+ -c.c.. Assume that $\langle p_i : i < \kappa^+ \rangle$ is a sequence of conditions. Pick $S \subseteq \kappa^+$ unbounded and a finite $b^* \subseteq \kappa$ such that for every $i \in S$, $p_i \setminus \max b^* + 1 \geq^* \dot{0}_{P \setminus \max b^* + 1}$. Denote $\beta = \max b^* + 1$, and, by shrinking $S \subseteq \kappa^+$, we can assume that for every $i \neq i'$ in S , $p_i \restriction \beta = p_{i'} \restriction \beta$. It follows that for such $i \neq i'$ in S , $p_i, p_{i'}$ are compatible.

□

We are now ready for the proof that P_κ additionally satisfies the κ -approximation property.

Theorem 6.6 *Let $P = P_\kappa$ be as in subsection 1.2.*

Assume that P does not add fresh subsets to ordinals of cofinality $\geq \kappa$. By the results of the current paper, this holds, for instance, if one assumes:

1. *There exists a stationary set $I \subseteq \kappa$ consisting of singulars, such that for every $\mu \in I$,*
 - *If $\gamma < \mu$ then $|P_\gamma| < \mu$.*
 - *$\Vdash_{P_\mu} \langle P_\kappa \setminus \mu \rangle$ is μ^{++} -closed.*
2. *In the case where the full-support iteration is taken, assume also that for every $\alpha < \kappa$, $\langle \dot{Q}_\alpha, \leq_{\dot{Q}_\alpha}, \leq_{\dot{Q}_\alpha}^* \rangle$ satisfies the following property: for every $x, y, z \in \dot{Q}_\alpha$, if $z \leq_{\dot{Q}_\alpha}^* x, y$ then x, y are $\leq^*$ -compatible.*

Let $G \subseteq P$ be generic over V . Then V has the κ -cover and approximation properties in $V[G]$.

Proof. The κ -cover property was proved in the previous theorem. Thus, we concentrate on the approximation property.

First, let us justify that the above assumptions suffice to prove that, for every λ with $\text{cof}(\lambda) \geq \kappa$, $P = P_\kappa$ does not add fresh unbounded subsets of λ . Indeed:

- For $\text{cof}(\lambda) = \kappa$, it suffices to prove that no fresh subsets are added to κ , by lemma 6.4 and the fact that P has the κ^+ -cover property (proved in Theorem 6.5). The fact that no fresh subsets are added to κ follows from Theorems 3.1, 4.5, 5.1.

- For $\text{cof}(\lambda) > \kappa$: if the nonstationary support is taken, this follows from Lemma 3.4. For the Easton support, this follows from Lemma 6.4 and the fact that for a regular μ , fresh subsets of μ are not added by a forcing of size $< \mu$ (this is proved, for example, in Corollary 4.8 in [4]). For the full support iteration, this follows from Lemma 5.3.

We proceed and prove the κ -approximation property. We may consider only sets of ordinals. Assume that α^* is an ordinal, and $f: \alpha^* \rightarrow 2$ is the characteristic function of a κ -approximated subset of α^* . Our goal is to prove that $f \in V$.

Assume by induction that for every $\alpha < \alpha^*$, $f \restriction \alpha \in V$. If $\text{cf}(\alpha^*) \geq \kappa$, then the fact that P does not add fresh subsets to ordinals of cofinality $\geq \kappa$ shows that $f \in V$. Thus assume that $\text{cf}(\alpha^*) < \kappa$. Denote $\xi^* = \text{cf}(\alpha^*)$ and fix an increasing, cofinal sequence $\langle \alpha_\xi: \xi < \xi^* \rangle$ in α^* .

Work in $V[G \restriction \xi^* + 1]$. We construct a $\leq^*$ -increasing sequence $\langle p_\xi: \xi < \xi^* \rangle$ in $P \setminus \xi^* + 1$, and an increasing, continuous sequence of ordinals $\langle \beta_\xi: \xi < \xi^* \rangle$ below κ . In limit steps, we have enough closure to take upper bounds. So we concentrate on successor steps. Assume that p_ξ has been constructed. We construct $p_{\xi+1}$. Let D be the dense open set of conditions in the forcing $P \setminus \xi^* + 1$, which decide $\tilde{f} \restriction \alpha_\xi$. Then, by applying lemma 2.3, there exists $\beta_{\xi+1} > \beta_\xi$ and $p_{\xi+1} \geq^* p_\xi$, such that for every $r \geq p_{\xi+1} \restriction \beta_{\xi+1}$, there exists $r' \geq r$ such that $r' \restriction p_{\xi+1} \setminus \beta_{\xi+1} \in D$. This concludes the successor step.

Let p^* be an upper bound of the sequence $\langle p_\xi: \xi < \xi^* \rangle$. Let $\beta^* = \sup\{\beta_\xi: \xi < \xi^*\}$. Then, over $V[G_{\xi^*+1}]$, the following property holds: for every $\xi < \xi^*$, and $r \geq p^* \restriction \beta^*$ there exists $r' \geq r$ such that $r' \restriction p^* \setminus \beta^*$ decides $\tilde{f} \restriction \alpha_\xi$.

Clearly, we can assume that the same property holds over V rather than $V[G_{\xi^*+1}]$. So we can assume that $p^* \in P_\kappa$.

Now, we finish the proof by integrating an argument of Hamkins from [8]. Let $T \subseteq 2^{<\alpha^*}$ be a sub-tree of the full binary tree, consisting of all the sequences $\sigma \in 2^{<\alpha^*}$ for which there exists an extension of p^* which forces that $\tilde{f} \restriction \text{lh}(\sigma) = \sigma$.

For every $\sigma \in T$, let p_σ be an arbitrary condition which extends p^* and forces $\tilde{f} \restriction \text{lh}(\sigma) = \sigma$. Note that we do not require that the p_σ -s extend each other; the only requirement is $p_\sigma \geq p^*$.

Fix $\sigma \in T$. Note that p_σ extends the condition $r \restriction p^* \setminus \beta^*$ for $r = p_\sigma \restriction \beta^*$. Let $r' \geq r$ be a condition such that $r' \restriction p^* \setminus \beta^*$ decides $\tilde{f} \restriction \text{lh}(\sigma)$. Denote $p_\sigma^* = r' \restriction p^* \setminus \beta^*$. Then p_σ^*, p_σ are compatible, and both decide $\tilde{f} \restriction \text{lh}(\sigma)$. Thus, the decided value is the same.

So we have constructed a sub-tree T of the binary tree, and an associated tree of conditions $\langle p_\sigma^*: \sigma \in T \rangle$, such that each condition p_σ^* has the form $r \restriction p^*$ (for some $r \in P_{\beta^*}$), and

$p_\sigma^* \Vdash \widetilde{f} \restriction \text{lh}(\sigma) = \sigma$.

Let $X \subseteq T$ be the set of splitting nodes, namely nodes $\sigma \in T$ such that $\sigma \smallfrown \langle 0 \rangle, \sigma \smallfrown \langle 1 \rangle \in T$.

Claim 5 $|X| < \kappa$.

Proof. For every $\sigma \in X$, $p_{\sigma \smallfrown \langle 0 \rangle}^*$ and $p_{\sigma \smallfrown \langle 1 \rangle}^*$ are defined, and have the form $r_{\sigma \smallfrown \langle 0 \rangle} \smallfrown p^* \setminus \beta^*$ and $r_{\sigma \smallfrown \langle 1 \rangle} \smallfrown p^* \setminus \beta$ for some $r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle} \in P_{\beta^*}$. Note that $r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}$ are incompatible.

Now, in order to prove that $|X| < \kappa$, consider the map $\sigma \mapsto \{r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}\}$ from X to $[P_{\beta^*}]^2$. Since $|(P_{\beta^*})^2| < \kappa$, it suffices to prove that this map is injective.

Indeed, assume that $\sigma \neq \sigma'$ are distinct points of X , which are mapped to the pairs $\{r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}\}$ and $\{r_{\sigma' \smallfrown \langle 0 \rangle}, r_{\sigma' \smallfrown \langle 1 \rangle}\}$, respectively.

If σ, σ' are incompatible, then $\{r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}\} \neq \{r_{\sigma' \smallfrown \langle 0 \rangle}, r_{\sigma' \smallfrown \langle 1 \rangle}\}$, since the conditions in the former pair, concatenated with $p^* \setminus \beta^*$, force that $\widetilde{f} \restriction (\text{lh}(\sigma)) = \sigma$, while the conditions in the latter pair, concatenated with $p^* \setminus \beta^*$, force that $\widetilde{f} \restriction (\text{lh}(\sigma')) = \sigma'$.

Thus, assume that σ, σ' are compatible, and, without loss of generality, σ' strictly extends σ , and $\sigma'(\text{lh}(\sigma)) = 0$. Then any condition in the pair $\{r_{\sigma' \smallfrown \langle 0 \rangle}, r_{\sigma' \smallfrown \langle 1 \rangle}\}$ forces, when concatenated with $p^* \setminus \beta^*$, that $\widetilde{f} \restriction (\text{lh}(\sigma) + 1) = \sigma \smallfrown \langle 0 \rangle$. On the other hand, the condition $r_{\sigma \smallfrown \langle 1 \rangle}$, which belongs to the pair $\{r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}\}$, forces, when concatenated with $p^* \setminus \beta^*$, that $\widetilde{f} \restriction (\text{lh}(\sigma) + 1) \neq \sigma \smallfrown \langle 0 \rangle$. It follows that $\{r_{\sigma \smallfrown \langle 0 \rangle}, r_{\sigma \smallfrown \langle 1 \rangle}\} \neq \{r_{\sigma' \smallfrown \langle 0 \rangle}, r_{\sigma' \smallfrown \langle 1 \rangle}\}$, as desired. $\square$

Finally, since f is κ -approximated and $|X| < \kappa$, $f \restriction X \in V$. Take any extension p^{**} of p^* which decides $\widetilde{f} \restriction X$. Denote by $g: X \rightarrow 2$ the function in V for which $p^{**} \Vdash \widetilde{f} \restriction X = g$. Then $p^{**} \Vdash \widetilde{f} = \bigcup \{\sigma \in T: \sigma \restriction X \subseteq g\}$, namely p^{**} forces that $\widetilde{f} \in V$. $\square$

We conclude this paper with an affirmative answer to question 1.4.

Corollary 6.7 *Assume that P_κ is an iteration of Prikry-type forcings as in the previous theorem, and $G \subseteq P_\kappa$ is generic over V . Let $W \in V[G]$ be a κ -complete ultrafilter whose underlying set belongs to X . Then $W \cap V \in V$.*

Proof. This is an immediate consequence of Theorem 6.6 and Lemma 6.2.

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